Hazard Assessment for Pyroclastic Flows

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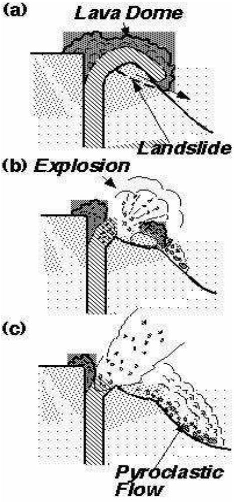
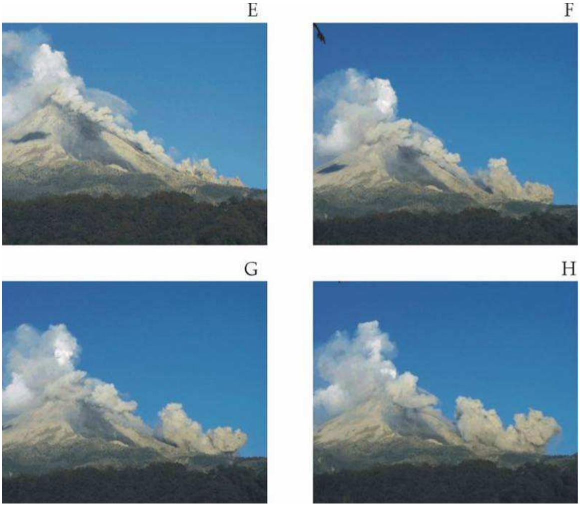
Games & Decisions in Reliability & Risk:
2011 May 21, Laggo Maggiore, IT

The Soufrière Hills Volcano on Montserrat

Pyroclastic Flows

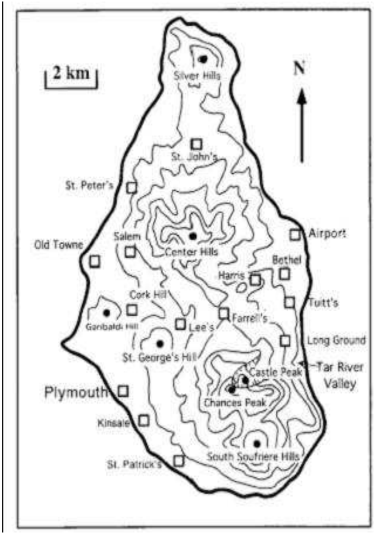


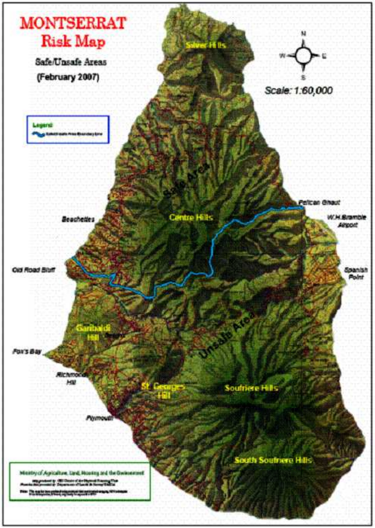


Pyroclastic Flows at Soufrière Hills Volcano

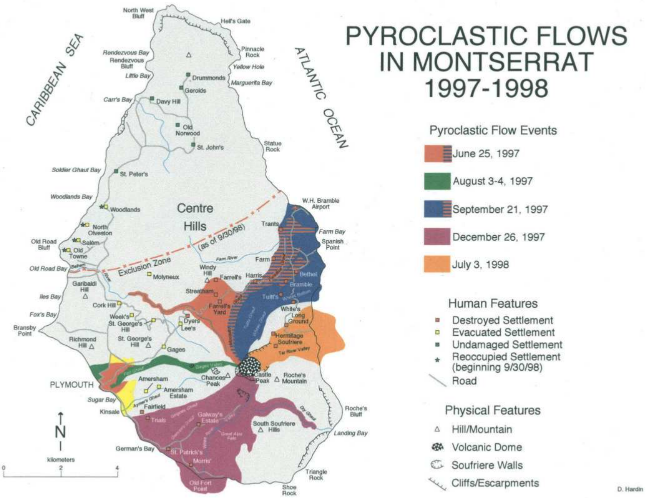
Where *is* Montserrat?

What's there?


2 km
N
Silver Hills
St. John's
St. Peter's
Airport
Old Towne
Salem
Center Hills
Bethel
Harris
Cork Hill
Tuitt's
Farrell's
Garibaldi Hill
Lee's
Long Ground
St. George's Hill
Castle Peak
Tar River Valley
Plymouth
Chances Peak
Kinsale
St. Patrick's
South Soufriere Hills


MONTSERRAT
Risk Map
Safe/Unsafe Areas
(February 2007)
Scale: 1:60,000
Silver Hills
Safe Area
Centre Hills
Beachettes
Old Road Bluff
Garibaldi Hill
Fox's Bay
Unsafe Area
St Georges Hill
Soufriere Hills
Plymouth
South Soufriere Hills
Spanish Point
W.H.Bramble Airport

Montserrat PFs in One 13-month Period



Plymouth, the former capital

Modelling Hazard

- Goal: quantify the hazard from Pyroclastic Flows (PFs)—
- For specific locations "x" on Montserrat, we would like to find the probability of a "catastrophic event" (inundation to ≥1m) within T years— for $T = 1$, $T = 5$, $T = 10$, $T = 20$ years
- Reflecting all that is uncertain, including:
  - How often will PFs of various volumes occur?
  - What initial direction will they go?
  - How will the flow evolve?
  - How are things changing, over time?
- This is *not* what is needed for short-term crisis and event management— here we consider only long-term hazard.

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Pieces of the Puzzle

Our team (geologist/volcanologist, applied mathematicians, statisticians, stochastic processes) breaks the problem down into two parts:

- Try to model the volume, frequency, and initial direction of PFs at SHV, based on 15 years' data;
- Tools used: Probability theory, extreme value statistics.
- For specific volume V and initial direction θ (and other uncertain things like the basal friction angle ϕ), try to model the evolution of the PF, and max depth M_x at location x;
- Tools used: Titan2D PDE solver, emulator (see below).

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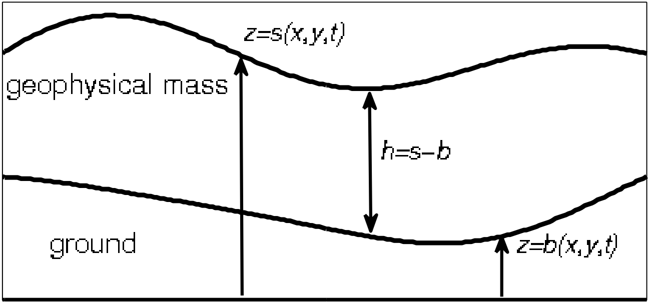
Titan2D



- Use 'thin layer' modeling $\rightsquigarrow$ system of PDEs for flow depth and depth-averaged momenta
- Incorporates topographical data from DEM
- Consumes 20m–2hr of computing time (in parallel on 16 processors) for each run— say, about 1 hr.

Titan2D (cont)

TITAN2D (U Buffalo) computes solution to the PDEs with:

- Stochastic inputs whose randomness is the basis of the hazard uncertainty:

  $V =$ initial volume (initial flow magnitude, in m^3),
  $\theta =$ initial angle (initial flow direction, in radians).

- Deterministic inputs:

$\phi =$	basal friction	(deg)	important & very uncertain
$\psi =$	internal friction	(deg)	less important, ignored
$v_0 =$	initial velocity	(m/s)	less important, set to zero

- Output: flow height and depth-averaged velocity at each of thousands of grid points at every time step. We will focus on the maximum flow height at a few selected grid points.
- Each run takes about 1 hour on 16 processors

Hazard Assessment I: What's a Catastrophe?

- Let $M_x(z)$ be the computer model prediction with arbitrary input parameter vector $z \in \mathcal{Z}$ of whatever characterizes a catastrophe at $x \in \mathcal{X}$.

SHV: *$z = (V, \theta, \phi) \in \mathcal{Z} = (0, \infty) \times [0, 2\pi) \times (0, 90)$, $M_x(V, \theta, \phi)$ = max PF height at location x (downtown Plymouth, Bramble Airport, Dyers RV TP4) for PF of characteristics $z = (V, \theta, \phi)$.*

- Catastrophe occurs for z such that $M_x(z) \in \mathcal{Y}_C$.

SHV: *Catastrophe if $M_x(z) \geq 1\, m$ (Other options...)*

- Determine 'catastrophic region' $\mathcal{Z}_c$ in the input space:

$$\mathcal{Z}_c = \{z \in \mathcal{Z} : \; M_x(z) \in \mathcal{Y}_C\}$$

SHV: *$\mathcal{Z}_c = \{z : \; V > \Psi(\theta, \phi)\}$ where...*

Which inputs are catastrophic at SHV?

By continuity and monotonicity,

$$\begin{aligned} \mathcal{Z}_c &= \{z \in \mathcal{Z} : \; M_x(z) \in \mathcal{Y}_C\} \\ &= \{(V, \theta, \phi) \in \mathcal{Z} \; : \; V > \Psi(\theta, \phi)\} \end{aligned}$$

for the *critical contour* Ψ, where

$$\Psi(\theta, \phi) \equiv \{\text{value of } V \text{ such that } M_x(V, \theta, \phi) = 1\,\text{m}\}$$

Now let's turn to the problem of evaluating Ψ.

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Emulation

To find Ψ, we'd like to evaluate $M_x(V_i, \theta_i, \phi_i)$ by running Titan2D for selected locations $x \in \mathcal{X}$ and at each of perhaps:

- ~ 100 Volumes V_i;
- ~ 100 Initiation Angles θ_j;
- ~ 100 Basal Friction Angles ϕ_k;

Which would entail maybe $100 \times 100 \times 100 = 1\,000\,000$ runs of Titan2D... but we don't have $1\,000\,000$ hours.

Our Solution:

Build an Emulator for our PDE flow model.

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Emulators

An Emulator is a (very fast):

- statistical model (based on Gaussian Processes) for our
- computer model (based on PDE solver) of the
- volcano.

- The emulator can predict (in seconds) what Titan2D would return (in hours), *with* an estimate of its accuracy;
- Based on Gaussian Stochastic Process (GaSP) Model. We:
  - Pick a few hundred LHC "design points" $(V_j, \theta_j, \phi_j, \ldots)$;
  - Run Titan2D at each of them to find Output $M_x(V_j, \theta_j, \phi_j)$;
  - Train the GaSP to return a statistical estimate

$$\mathsf{E}\big[M_x(V,\theta,\phi) \mid \{M_x(V_j,\theta_j,\phi_j)\}\big]$$

of model output M for site x at untried points (V, θ, ϕ).

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Why?

Our immediate goal is to find a "threshhold function" for each location of concern x in Montserrat:

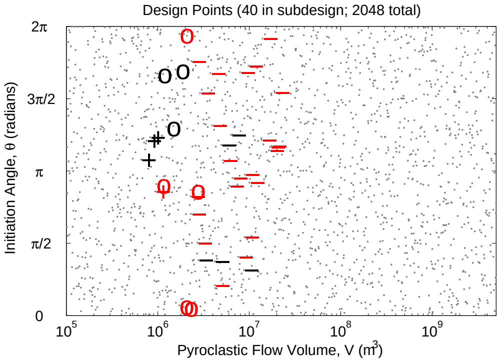
$$\Psi(\theta,\phi) = \text{Smallest volume } V \text{ that would inundate } x \text{ if flow begins in direction } \theta \text{ with friction angle } \phi$$

for each possible direction θ (0–360 in degrees, or 0–2π radians, with $0=$ due East and $\pi/2=$ due North) and basal friction angle ϕ.
Simplification: Take $\phi_i \equiv \hat{\phi}(V_i)$ (empirical; see below).
Then we can quantify the hazard at x for T years as

$$\Pr\left\{V_i \geq \Psi(\theta_i) \text{ for some PF } (V_i,\theta_i) \text{ in time } (t_0, t_0+T]\right\}$$

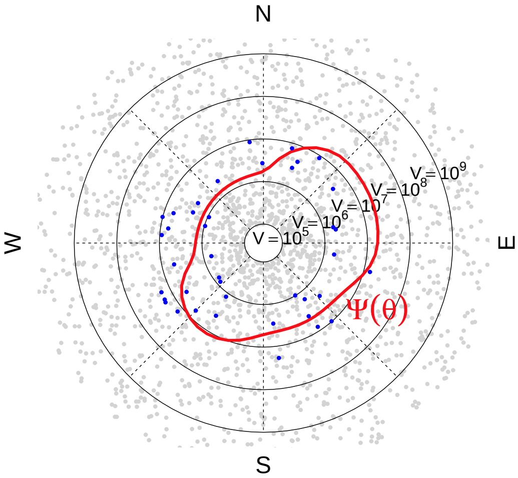
which in turn we study with the probability models.

Design points V, θ

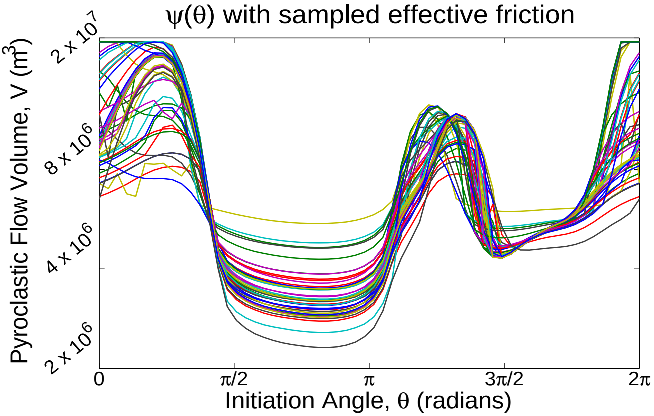


2048 Design Points total; 40 used for determining Ψ at $x=$ Level 4 Trigger Point: Dyers River Valley (head of Belham valley) .
Black $\Longleftrightarrow M_x(V, \theta, \phi) = 0$, Red $\Longleftrightarrow M_x(V, \theta, \phi) > 0$.

Polar View of Design, Subdesign, & Ψ



50 Simulations of $\Psi(\theta)$ for Level 4 Trigger Point, Dyers RV



Some Details about our Emulator...

Emulators are very fast *approximations* for (some of) the outputs $M_x(z)$ of (slow) computer models. They are used for many purposes (design, optimization, inference, sensitivity analyses, ...) for expensive computer models.

- Begin with a Maxi-Min Latin Hypercube statistical design to select some number N of design points z_i in the large region $\mathcal{Z} = [10^5, 10^{9.5}]\mathrm{m}^3 \times [0, 2\pi)\mathrm{rad} \times [5, 25)\mathrm{deg}$.
- Run the slow computer model $M_x(z_i)$ at these N preliminary points.
- For fitting an emulator to find $\mathcal{Z}_c$, keep only design points $\mathcal{D}$ in a region 'close' to the boundary $\partial\mathcal{Z}_c$:

Too many details? Skip Emulator Details

Gaussian Process Emulators in the Region $\mathcal{Z}_c$

- Since we are interested in regions where the flow is small (1m), fit an emulator to $\tilde{M}_x(z) \equiv \log(1 + M_x(z))$. Let $\widetilde{\mathbf{y}}$ be the transformed vector of computer model runs $\tilde{M}_x(z)$ for $z \in \mathcal{D}$.
- Model the unknown $\tilde{M}(z)$ as a Gaussian process

$$\tilde{M}(z) = \beta + mV + Z(V, \theta, \phi)$$

(note that we expect a monotonic trend in V, but not θ; ϕ discussed later), where $Z(V, \theta, \phi)$ is a stationary GP with

  - Mean 0, Variance σ_z^2;
  - Product exponential correlation: *i.e.*, $(\forall z_i = (V_i, \theta_i, \phi_i) \in \mathcal{D})$, the correlation matrix $\mathbf{R}$ is:

$$R_{ij} = \exp\left\{ - \left| \frac{V_i - V_j}{\rho_V} \right|^{\alpha_V} - \left| \frac{\theta_i - \theta_j}{\rho_\theta} \right|^{\alpha_\theta} - \left| \frac{\phi_i - \phi_j}{\rho_\phi} \right|^{\alpha_\phi} \right\}$$

with range parameters $\rho_\bullet$, smoothness parameters $\alpha_\bullet$.

Handling the Unknown Hyperparameters $\boldsymbol{\theta} = (\rho_V, \rho_\theta, \rho_\phi, \alpha_V, \alpha_\theta, \alpha_\phi, \sigma_z^2, \beta, m)$

- Deal with the crucial parameters (σ_z^2, β, m) via a fully Bayesian analysis (here an extension of Kriging) using objective priors: $\pi(\beta) \propto 1$, $\pi(m) \propto 1$, and $\pi(\sigma_z^2) \propto 1/\sigma_z^2$;
- Compute the marginal posterior mode, $\hat{\boldsymbol{\xi}}$, of $\boldsymbol{\xi} = (\rho_V, \rho_\theta, \rho_\phi, \alpha_V, \alpha_\theta, \alpha_\phi)$ using the above priors and the reference prior for $\boldsymbol{\xi}$; then $\hat{\mathbf{R}} \equiv \mathbf{R}(\hat{\boldsymbol{\xi}})$ is completely specified (big simplification— no matrix decomp inside MCMC loop).
    - A fully Bayesian analysis, accounting for uncertainty in $\hat{\boldsymbol{\xi}}$, is difficult and rarely affects the final answer significantly because of confounding of variables.

The Posterior Mode of $\boldsymbol{\xi} = (\rho_V, \rho_\theta, \rho_\phi, \alpha_V, \alpha_\theta, \alpha_\phi)$

- MLE fitting of $\boldsymbol{\xi}$ has enormous problems; we've given up on it.
- A big improvement is finding the *marginal* MLE of $\boldsymbol{\xi}$ from the marginal likelihood for $\boldsymbol{\xi}$, available by integrating lh wrt the objective prior $\pi(\beta, m, \sigma_z^2) = 1/\sigma_z^2$. The expression is:

$$L(\boldsymbol{\xi}) \propto |\mathbf{R}(\boldsymbol{\xi})|^{-\frac{1}{2}} \left|\mathbf{X}'\mathbf{R}(\boldsymbol{\xi})^{-1}\mathbf{X}\right|^{-\frac{1}{2}} (S^2(\boldsymbol{\xi}))^{-(\frac{n-q}{2})}, \text{ where}$$

  - $\mathbf{X} = [\mathbf{1}, \mathbf{V}]$ is the design matrix for the linear parameters, *i.e.*, $\mathbf{1}$ is the column vector of ones and $\mathbf{V}$ is the vector of volumes $\{V_i\}$ in the data set, and $\boldsymbol{\mu} = (\beta, m)$ (of dimension $q = 2$);
  - $S^2(\boldsymbol{\xi}) = [\widetilde{\mathbf{y}} - \mathbf{X}\hat{\boldsymbol{\mu}}]'\mathbf{R}(\boldsymbol{\xi})^{-1}[\widetilde{\mathbf{y}} - \mathbf{X}\hat{\boldsymbol{\mu}}]$;
  - $\hat{\boldsymbol{\mu}} = [\mathbf{X}'\mathbf{R}(\boldsymbol{\xi})^{-1}\mathbf{X}]^{-1}\mathbf{R}(\boldsymbol{\xi})^{-1}\widetilde{\mathbf{y}}$.

- An even bigger improvement arises by finding the posterior mode from $L(\boldsymbol{\xi})\pi^R(\boldsymbol{\xi})$, where $\pi^R(\boldsymbol{\xi})$ is the reference prior for $\boldsymbol{\xi}$ (Paulo, 2005 AoS). Note that it is computationally expensive to work with the reference posterior in an MCMC loop, but using it for a single maximization to determine the posterior mode is cheap.
The reference prior for $\boldsymbol{\xi}$ is $\pi^R(\boldsymbol{\xi}) \propto |I^\star(\boldsymbol{\xi})|^{1/2}$, where

$$I^\star(\boldsymbol{\xi}) = \begin{pmatrix} (n-q) & \mathrm{tr}\mathbf{W}_1 & \mathrm{tr}\mathbf{W}_2 & \cdots & \mathrm{tr}\mathbf{W}_p \\ & \mathrm{tr}\mathbf{W}_1^2 & \mathrm{tr}\mathbf{W}_1\mathbf{W}_2 & \cdots & \mathrm{tr}\mathbf{W}_1\mathbf{W}_p \\ & & \ddots & \cdots & \vdots \\ & & & & \mathrm{tr}\mathbf{W}_p^2 \end{pmatrix}$$

$$\mathbf{W}_k = \left(\frac{\partial}{\partial \xi_k}\right) \mathbf{R}(\xi)^{-1} \left\{ \mathbf{I}_n - \mathbf{X}\left[\mathbf{X}'\mathbf{R}\xi\right]^{-1}\mathbf{X} \right\}^{-1} \mathbf{X}'\mathbf{R}(\xi)^{-1},$$

with $q = 2$ the dimension of $\boldsymbol{\mu}$ and $p = 3$ the dimension of $\boldsymbol{\xi}$.

The posterior distribution of (σ_z^2, β, m), conditional on $\widetilde{\mathbf{y}}$ and $\hat{\boldsymbol{\xi}}$, yields the final emulator (in transformed space) at input z^*:

$$\tilde{M}_x(z^*) \mid \widetilde{\mathbf{y}},\ \hat{\boldsymbol{\xi}} \ \sim\ t(\, y^*(z^*),\, s^2(z^*),\, N-2)\,,$$

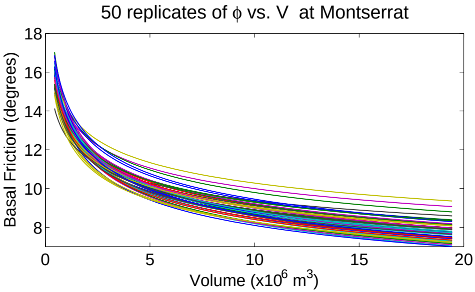
noncentral t-distribution with $N-2$ degrees of freedom and location & scale parameters:

$$y^*(z^*) = \mathbf{r}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}} + \frac{\mathbf{1}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}}}{\mathbf{1}^T\mathbf{R}^{-1}\mathbf{1}}[1-\mathbf{r}^T\mathbf{R}^{-1}\mathbf{1}] + \frac{\widetilde{\mathbf{V}}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}}}{\widetilde{\mathbf{V}}^T\mathbf{R}^{-1}\widetilde{\mathbf{V}}}[\widetilde{V}^* - \mathbf{r}^T\mathbf{R}^{-1}\widetilde{\mathbf{V}}]$$

$$s^2(z^*) = \left[(1-\mathbf{r}^T\mathbf{R}^{-1}\mathbf{r}) + \frac{(1-\mathbf{r}^T\mathbf{R}^{-1}\mathbf{1})^2}{\mathbf{1}^T\mathbf{R}^{-1}\mathbf{1}} + \frac{(\widetilde{V}^* - \mathbf{r}^T\mathbf{R}^{-1}\widetilde{\mathbf{V}})^2}{\widetilde{\mathbf{V}}^T\mathbf{R}^{-1}\widetilde{\mathbf{V}}}\right] \times \frac{1}{N-2}\left[\widetilde{\mathbf{y}}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}} - \frac{(\mathbf{1}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}})^2}{\mathbf{1}^T\mathbf{R}^{-1}\mathbf{1}} - \frac{(\widetilde{\mathbf{V}}^T\mathbf{R}^{-1}\widetilde{\mathbf{y}})^2}{\widetilde{\mathbf{V}}^T\mathbf{R}^{-1}\widetilde{\mathbf{V}}}\right],$$

where $\widetilde{V}_i = V_i - V_R$, $\widetilde{V}^* = V^* - V_R$, $V_R = \mathbf{1}^T\mathbf{R}^{-1}\mathbf{V}/\mathbf{1}^T\mathbf{R}^{-1}\mathbf{1}$, and $\mathbf{r}^T = (R(z^*, z_1), \ldots, R(z^*, z_N))$. Tedious, but tractable.

Theoretically the Basal Friction ϕ *should* be constant, but empirically it is highly dependent on the Volume V:



Currently we simply estimate the function $\phi(V)$, and replace ϕ in the emulator by this function. The emulator thus becomes only a function of (V, θ). We are now moving to a full 3-dimensional $\mathcal{Z}_c$.

With this short-cut, $\Psi(z)$ depends on only one quantity: θ.

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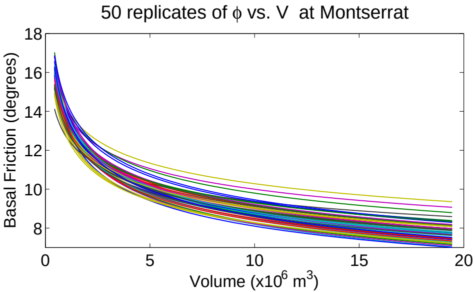

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Response Surfaces

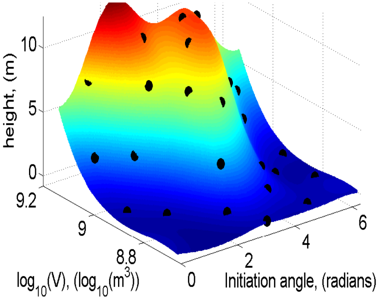

Figure: Median of the emulator, transformed back to the original space.
Left: Plymouth, Right: Bramble Airport.
Black points: max-height simulation outputs at subdesign points $\mathcal{D}_c$.

Catastrophic event contours Ψ (response surface slices at ht. 1 m)

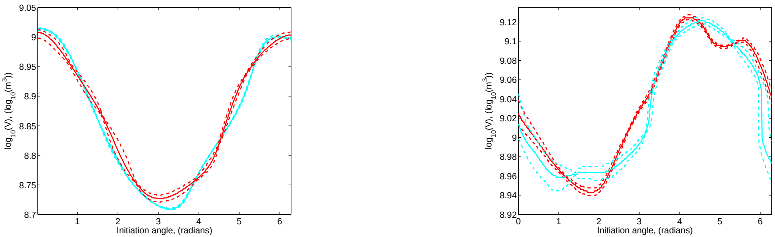

Figure: Left: Plymouth, Right: Bramble Airport.
Solid Cyan: linear max-height. Solid Red: log transformation.
Dashed: 75% pointwise confidence bands for Ψ.

Adapting the design

- We added new design points near the boundary $\partial\mathcal{Z}_c$ of the critical region where:
  - contours $\Psi(z)$ pass between design points z_i with $M_x(z_i) = 0$ and z_j with $M_x(z_j) \gg 1$; or
  - the confidence bands for $\Psi(z)$ are widest.
- The computer model was re-run at these new design points.
- The emulator was then re-fit and and critical contour Ψ was re-computed.
- Median contours Ψ did not change much, but confidence bands for Ψ were much narrower, so it was judged that no further adaptation was needed.

(For other adaptive designs see R.B. Gramacy *et al.* (ICML, 2004), P. Ranjan *et al.* (Technometrics, 2008), B.J. Williams *et al.* (Stat Sin, 2000))

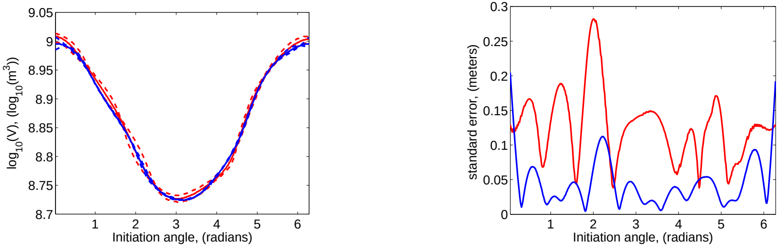

Figure: Left: $\Psi(\theta)$ (solid) with implicit standard error curves (dashed), Right: standard error; Red: original design. Blue: updated design.

The critical region $\mathcal{Z}_c$ of inputs (V, θ) producing a catastrophic event is the region above the critical contour Ψ.

Hazard Assessment II: Probability of Catastrophe

For us a PF is *catastrophic* if its volume V exceeds an uncertain threshold $\Psi(\theta)$ that depends on the initiation angle θ.
For a hazard summary we wish to report, for each $T > 0$,

$$\Pr[\text{ Catastrophe at } x \text{ within } T \text{ Years }] \\ = \Pr[\{(V_i, \theta_i, \tau_i)\} : \; V_i > \Psi(\theta_i), \; \tau_i \le T \;]$$

SO, we need a joint model for points $\{(V_i, \theta_i, \tau_i)\} \subset \mathbb{R}^3$.
Let's do it in that order: first Volumes, then Angles, then Times.

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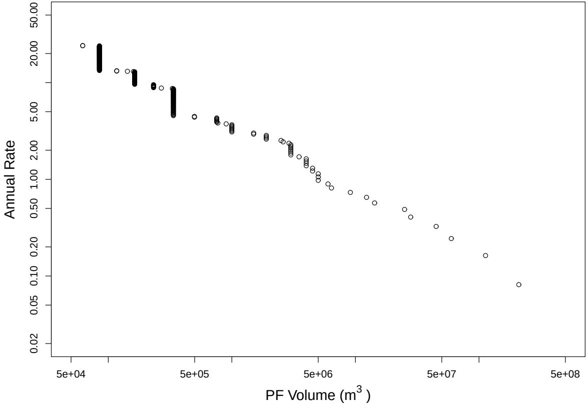
PF Volume *vs.* Frequency



Seems kind of linear, on log-log scale...

Linear log-log plots of Magnitude *vs.* Frequency are a hallmark of the Pareto probability distribution $\mathsf{Pa}(\alpha, \epsilon)$,

$$\mathsf{P}[V > v] = (v/\epsilon)^{-\alpha}, \qquad v > \epsilon.$$

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The Pareto Distribution

The negative slope seems to be about $\alpha \approx 0.64$ or so.
The Pareto distribution with $\alpha < 1$ has:

- Heavy tails;
- Infinite mean $\mathsf{E}[V] = \infty$, infinite variance $\mathsf{E}[V^2] = \infty$;
- No Central Limit Theorem for sums (skewed α-Stable);
- Significant chance that, in the future, we will see volumes larger than any we have seen in the past. Like $V > 10^9 \text{m}^3$.
- The Pareto comes up all the time in the Peaks over Threshhold (PoT) approach to the Statistics of Extreme Events— related to Fisher/Tippett Three Types Theorem.

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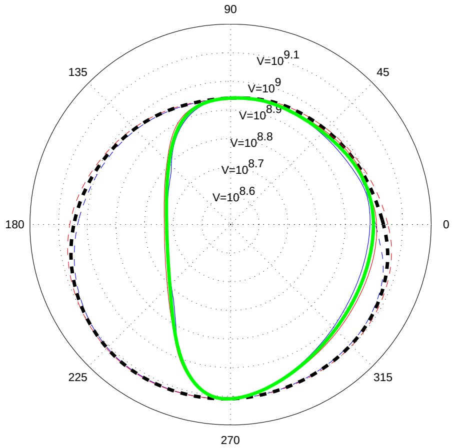
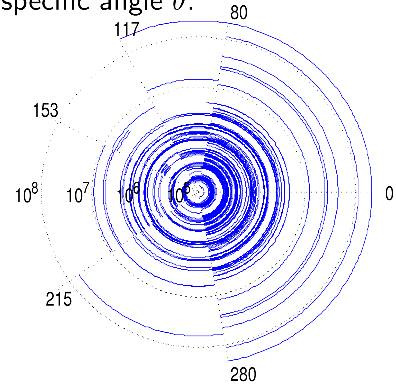
How big is bad?



Kinda depends on which direction it goes...

PF Initiation Angles

Our data on angles is quite vague— we only know which of 7 or 8 *valleys* were reached by a given PF, from which we can infer a *sector* but not a specific angle θ:



Any nonuniformity for θ? Any dependence of θ on Volume V?

Angle/Volume Data (cont)

We need a *joint* density function for V and θ. Without much evidence against independence, we use product pdf:

$$V, \theta \sim \alpha \epsilon^{\alpha} V^{-\alpha-1} \mathbf{1}_{\{V>\epsilon\}} \pi_{\kappa}(\theta)$$

where $\pi_{\kappa}(\theta)$ is the pdf for the von Mises vM(μ, κ) distribution,

$$\pi_{\kappa}(\theta) = \frac{e^{\kappa \cos(\theta-\mu)}}{2\pi I_0(\kappa)},$$

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PF Times

If we observe λ PFs per year of volume $V > \epsilon$, then what is the probability that such a PF will occur in the next 24 hours?

For *short-term* predictions, it may be important to note this can depend on many things, such as:

- How high is the dome just now?
- Any seismic activity suggesting dome growth and instability?
- Has it rained recently?
- How long since last PF?

But, for *long-term* predictions all these factors average out and we assume that:

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Under those assumptions (which we will revisit), the number of PFs in any time interval has a Poisson Distribution, with mean proportional to its length.
For an interval of length $\Delta t = 1/365$, a single day, the expected number of PFs is $\lambda \Delta t$ and so the *probability* of

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A Summary of our Stochastic Model:

The data suggest a (provisional) model in which:

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These have the beautiful and simplifying consequence that the number of PFs in *any* region of three-dimensional $(V \times \theta \times \tau)$ space has a Poisson probability distribution— so, we can evaluate:

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Hazard

$$
\begin{aligned}
&\mathsf{P}[\text{ Catastrophy within } T \text{ Years }] \\
&\qquad = 1 - \mathsf{P}[Y_T = 0] \qquad (\text{where } Y_T \text{ is the } \textit{number} \text{ of catastrophes}) \\
&\qquad = 1 - \exp\big(-\mathsf{E}Y_T \big) \\
&\qquad = 1 - \exp\left(-\lambda T \, \epsilon^\alpha \int_0^{2\pi} \Psi(\theta)^{-\alpha} \, \pi_\kappa(d\theta) \right)
\end{aligned}
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Which we can compute pretty easily on a computer.

Accommodating uncertainty in λ, κ, α, *etc.* is easy in Objective Bayesian statistics— we use Reference Prior distributions, and simulation-based methods (MCMC) to evaluate the necessary integrals.

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Posterior distribution of (α, λ)

For a given minimum volume $\epsilon > 0$ and period $(0, t]$, the sufficient statistics and Likelihood Function are

$$J = \text{Number of PF's in } (0, t], \text{ and}$$
$$S = \sum \log(V_j), \text{ the log-product of their volumes}$$
$$L(\alpha, \lambda) \propto (\lambda\,\alpha)^J \exp\left\{-\lambda t\,\epsilon^{-\alpha} - \alpha\,S\right\}$$

Objective Priors:

- Jeffreys prior is $\pi_J(\alpha, \lambda) \propto |I(\alpha, \lambda)|^{1/2} \propto \alpha^{-1}\epsilon^{-\alpha}$;
- Reference priors:
  - α of interest: $\pi_{R1}(\alpha, \lambda) \propto \lambda^{-1/2}\alpha^{-1}\epsilon^{-\alpha/2}$
  - λ of interest: $\pi_{R2}(\alpha, \lambda) \propto \lambda^{-1/2}[\alpha^{-2} + (\log\epsilon)^2]^{1/2}\epsilon^{-\alpha/2}$, which is also Jeffreys' independent prior.

Posterior: $\pi(\alpha, \lambda \mid \text{data}) \propto L(\alpha, \lambda)\,\pi(\alpha, \lambda)$, quite tractable.

Computing the probability of catastrophe

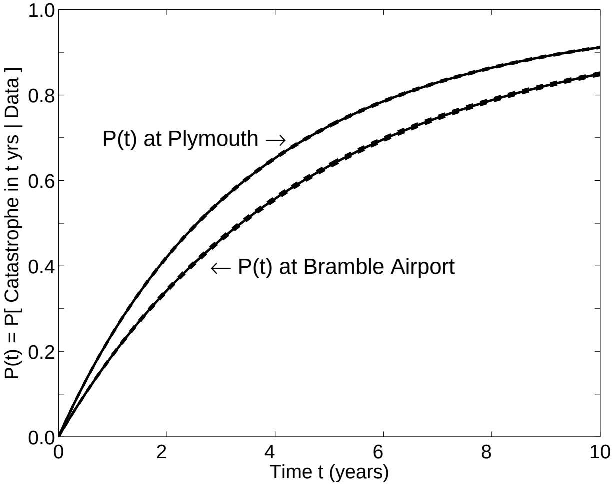
To compute Pr(at least one catastrophic event in T *years* | data) for a range of T, an importance sampling estimate is

$$P(T) \cong 1 - \frac{\sum_i \exp\left[-\lambda_i\, T\, \epsilon^{\alpha}\, \widehat{\Psi}(\alpha_i)\right] \frac{\pi^*(\alpha_i,\lambda_i)}{g(\alpha_i,\lambda_i)}}{\sum_i \frac{\pi^*(\alpha_i,\lambda_i)}{g(\alpha_i,\lambda_i)}},$$

where

- $\widehat{\Psi}(\alpha)$ is an MC estimate of $\int_0^{2\pi} \Psi(\theta)^{-\alpha}\, \pi_\kappa(d\theta)$ based on draws $\theta_i \sim \mathrm{vM}(\mu, \kappa)$ (or use quadrature);
- $\pi^*(\alpha, \lambda)$ is the un-normalized posterior density;
- $\{(\alpha_i, \lambda_i)\}$ are iid draws from the importance sampling density $g(\alpha, \lambda) = t_2(\alpha, \lambda \mid \widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}}, 3)$, with d.f. 3, mean $\widehat{\boldsymbol{\mu}}^t = (\hat{\alpha}, \hat{\lambda})$, and scale $\widehat{\boldsymbol{\Sigma}}$ = inverse of observed Information Matrix.

Hazard Over Time at Plymouth & Bramble



How are things at Bramble Airport?

Not so good... here's the runway...

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Discussion

We have argued that:

- Hazard assessment of catastrophic events (in the absence of lots of extreme data) requires
  - Mathematical computer modeling to support extrapolation beyond the range of the data;
  - Statistical modeling of available (possibly not extreme) data to determine input distributions;
  - Statistical development of emulators of the computer model to determine critical event contours.
- Major sources of uncertainty can be combined and incorporated with Objective Bayesian analysis.

Ongoing Work & Extensions:

- Extend the methodology to create entire Hazard Maps (*i.e.*, find hazard for all locations x simultaneously).
- Go beyond stationarity with
  - Change-point model for intensity λ_t;
  - Model (heavy-tailed) duration of activity;
  - Model caldera evolution (μ, κ for vM).
- Reflect uncertainty and change in DEMs.

A Collaborative Effort...

Thanks— to Organizers and Collaborators!

For more, see: www.stat.duke.edu/∼rlw/ or www.mvo.ms