Nonparametric Bayesian Spatial Statistics

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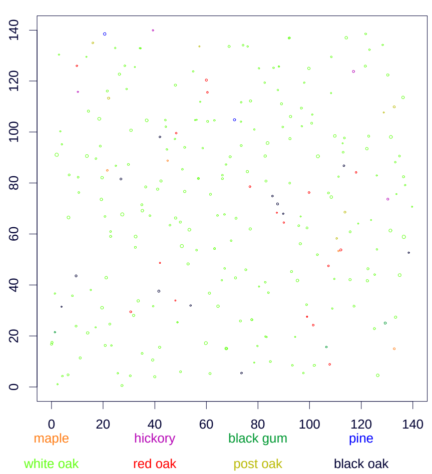
3. Hierarchical Models and Bayesian Computation

- I.I.D. Poisson Models
- Exchangeable Poisson Models
- Spatial Poisson Models
- Continuous Spatial Poisson Models
- Example: Biodiversity in Duke Forest

Overstory Trees ($D > 25cm$) in Bormann Plot



Big Oak Trees: How many per 20×20m square?

- Guess: 2? 5? 10?
- Density: $f(\lambda) \propto \lambda^{\alpha} e^{-\tau\lambda}$
- Mean: $E[\lambda] = \alpha/\tau \approx 5$
- Variance: $V[\lambda] = \alpha/\tau^2 \approx 25$
- $\Rightarrow \alpha = 1, \; \tau = 0.20$:

Now $\lambda \sim \mathsf{Ga}(\alpha, \tau)$ has "prior" density $f(\lambda) \propto \lambda^\alpha e^{-\tau\lambda}$ and, conditional on λ, $\{x_i\}$ are independent $\mathsf{Po}(\lambda)$; the joint density is then

$$\begin{aligned} f(\lambda, x_1, ..., x_n) &= \frac{\tau^\alpha \lambda^\alpha e^{-\tau\lambda}}{\Gamma(\alpha)} \times \prod \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \\ &= c\, \lambda^{\alpha+\Sigma x_j - 1}\, e^{-(\tau+n)\lambda} \end{aligned}$$

so, **conditional** on $x_1, ..., x_n$, the **posterior** distribution of λ is $\lambda \sim \mathsf{Ga}(\alpha + \Sigma x_i, \tau + n)$ with mean and variance

$$\begin{aligned} E[\lambda | x_1, ..., x_n] &= \frac{\alpha + \Sigma x_i}{\tau + n} = p\frac{\alpha}{\tau} + q\frac{\Sigma x_i}{n} \\ V[\lambda | x_1, ..., x_n] &= \frac{\alpha + \Sigma x_i}{(\tau + n)^2} \end{aligned}$$

where $p = \frac{\tau}{\tau+n}$, $q = \frac{n}{\tau+n} = 1 - p$.

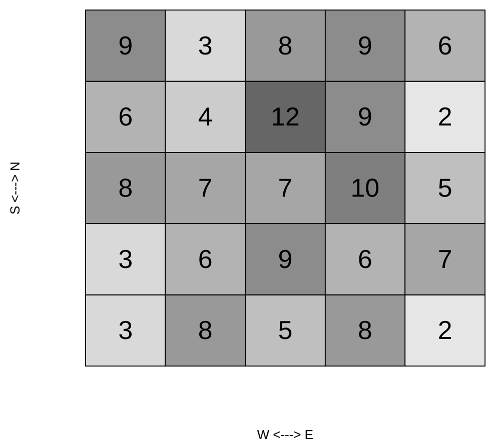
Large Oaks: Data from the Bormann Plot

9	3	8	9	6
6	4	12	9	2
8	7	7	10	5
3	6	9	6	7
3	8	5	8	2

Table 1: Oak tree counts in 20×20 quadrats

- Total: $\Sigma x_i = 162$
- Mean: $\bar{x} = 162/25 = 6.48$
- Variance: $s^2 = 6.926667 \Rightarrow s = 2.631856$

Oaks Distribution, Raw Data:



Oaks: BUGS Program for IID Oak Model

```
model {
 for(i in 1:N) {
   for(j in 1:N) {
     x[i,j] ~ dpois(lambda);
   }
 }
 lambda ~ dgamma(alpha,tau); }
data:
   list(alpha=1.0, tau=0.20, N=5, M=5,
      x=structure(.Data=c(9,3,8,9,6, 6,4,12,9,2,
      8,7,7,10,5, 3,6,9,6,7, 3,8,5,8,2),
      .Dim=c(5,5)));
init:
   list(lambda=1.0);
```

IID Oaks Results:

BUGS reports mean $\mu_\lambda = 6.466$ with standard deviation $\sigma_\lambda = 0.5065$ after 100 000 iterations, with an estimated Monte Carlo sampling error of ± 0.00166. The exact solution (from above) is

$$E[\lambda|x_1,...,x_n] = \frac{\alpha+\Sigma x_i}{\tau+n} = \frac{1+162}{0.20+25} = 6.468254$$

$$V[\lambda|x_1,...,x_n] = \frac{\alpha+\Sigma x_i}{(\tau+n)^2} = \frac{1+162}{(0.20+25)^2} = 0.506633^2$$

Evidently MCMC simulation works well for this easy problem.

Exchangeable Oaks Model

The simplest generalization of conjugate Poisson-gamma models that can reflect possible positive association among regions is to allow the Poisson means to be independent gamma variates, drawn from a common distribution about which we will learn from the data. We use the following three-stage hierarchical generalized model:

- Top: $\theta \sim \pi(\theta)$
- Mid: $\lambda_i \sim \mathsf{Ga}(\alpha^\theta, \tau^\theta)$
- Low: $X_i \overset{\text{ind}}{\sim} \mathsf{Po}(\lambda_i)$

We will choose $\pi(\theta)$ such that $\alpha^\theta \sim \mathsf{Ga}(2,2)$ and $\tau^\theta \equiv 0.2$.

Oaks: BUGS Program for Exchangeable Oak Model

```
model {
 for(i in 1:N) { for(j in 1:N) {
    x[i,j] ~ dpois(lam[i,j]);
    lam[i,j] ~ dgamma(alpha,tau);
  } }
 alpha ~ dgamma(2,2); }
data:
  list(tau=0.20, N=5, M=5,
    x=structure(.Data=c(9,3,8,9,6, 6,4,12,9,2,
    8,7,7,10,5, 3,6,9,6,7, 3,8,5,8,2), .Dim=c(5,5)));
init:
  list(alpha=1.0,
    lam=structure(.Data=c(1,1,1,1,1, 1,1,1,1,1,
    1,1,1,1,1, 1,1,1,1,1, 1,1,1,1,1), .Dim=c(5,5)));
```


Exchangeable Oaks Results:

9→8.82	3→3.84	8→8.01	9→8.84	6→6.35
6→6.34	4→4.67	12→11.34	9→8.84	2→3.01
8→7.99	7→7.16	7→7.16	10→9.67	5→5.50
3→3.83	6→6.34	9→8.84	6→6.34	7→7.18
3→3.85	8→7.99	5→5.50	8→7.99	2→3.00

Table 2: Exchangeable Oak posterior means in 20×20 quadrats

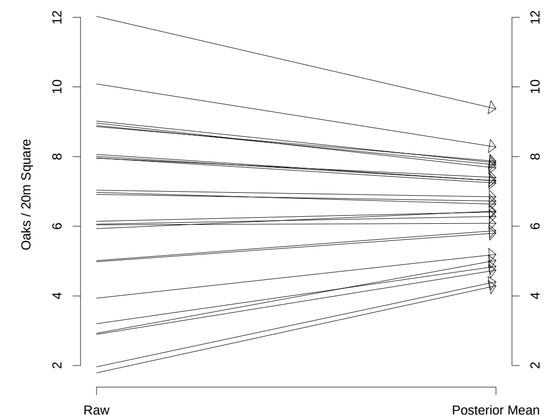
Oaks Results for Exchangeable Model:

S <---> N

8.8	3.8	8	8.8	6.4
6.3	4.7	11.3	8.8	3
8	7.2	7.2	9.7	5.5
3.8	6.3	8.8	6.3	7.2
3.8	8	5.5	8	3

W <---> E

Oaks Results:



Note uniform shrinkage

Spatial Oaks Model

The simplest generalization of conjugate Poisson-gamma models that can reflect possible positive association for neighboring regions is to allow the Poisson means to be, not independent gamma variates, but **linear combinations** of a collection of independent gamma variates, with overlapping linear combinations leading to dependence. We use the following three-stage hierarchical generalized model:

- Top: $\theta \sim \pi(\theta)$
- Mid: $\lambda_i = \sum k_{ij}^{\theta} \Gamma_j, \qquad \Gamma_j \sim \mathsf{Ga}(\alpha_j^{\theta}, \tau_j^{\theta})$
- Low: $X_i \overset{\text{ind}}{\sim} \mathsf{Po}(\lambda_i)$

with, for example, $k_{ij}^{\theta} = \theta_1$ for $i = j$, $k_{ij}^{\theta} = \theta_2$ for $i \sim j$, and otherwise $k_{ij}^{\theta} = 0$.

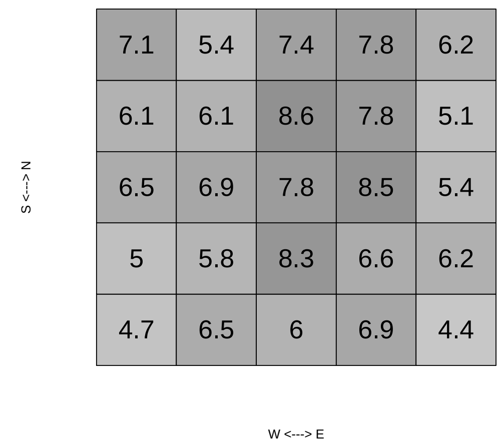
$$\begin{array}{llcl}
\text{Parameter:} & \theta & \sim & \pi(\theta) \\
\text{Impulses:} & \{\Gamma_j\}_{j\in J}|_{\theta} & \overset{\text{ind}}{\sim} & \mathsf{Ga}(\alpha_j^{\theta}, \tau_h^{\theta}) \\
\text{Intensities:} & \Lambda_i & \equiv & \sum_{j\in J} k_{ij}^{\theta}\Gamma_j \\
\text{Point counts:} & \{N_i\}_{i\in I} & \overset{\text{ind}}{\sim} & \mathsf{Po}(\Lambda_i)
\end{array}$$

Oaks: Posterior Mean of Oak Data

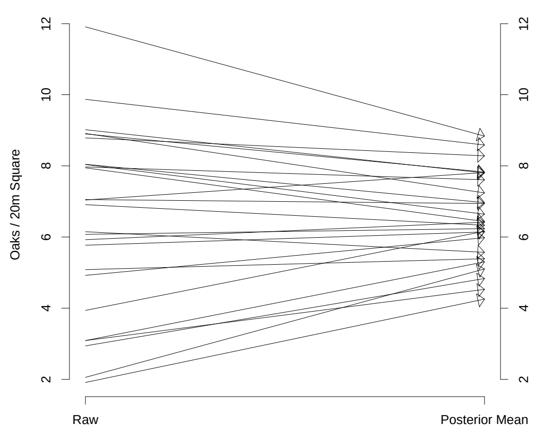
7.12	5.35	7.45	7.75	6.17
6.08	6.08	8.64	7.82	5.06
6.54	6.90	7.80	8.49	5.43
4.97	5.79	8.27	6.55	6.24
4.70	6.54	5.95	6.93	4.38

Table 3: Posterior Means, Spatial Model

Oaks Results for Spatial Model:



Spatial Oaks Results:



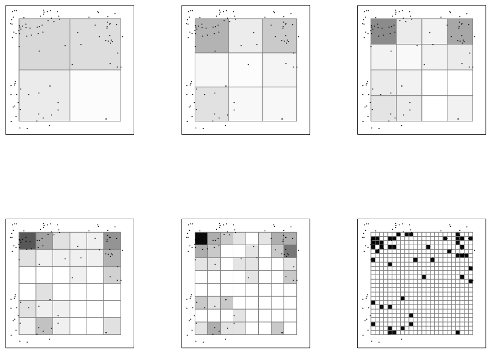
Note NON-uniform shrinkage

The **mean vectors** and **covariance matrices** of the point counts, conditional on $\theta \in \Theta$, follow by routine computation:

$$\begin{aligned} \mathsf{E}^\theta\left[N_i\right] &= \sum_{j\in J} k^\theta_{ij}(\tau^\theta_j)^{-1}\alpha^\theta_j \\ \mathsf{Cov}^\theta\left[N_i, N_{i'}\right] &= \sum_{j\in J} \left(\delta^i_{i'} + k^\theta_{i'j}(\tau^\theta_j)^{-1}\right) k^\theta_{ij}(\tau^\theta_j)^{-1}\alpha^\theta_j \end{aligned}$$

Unconditionally the $\{N_i\}_{i\in I}$ are distributed as sums of independent negative-binomial random variables.

What Level of Aggregation Should We Use?

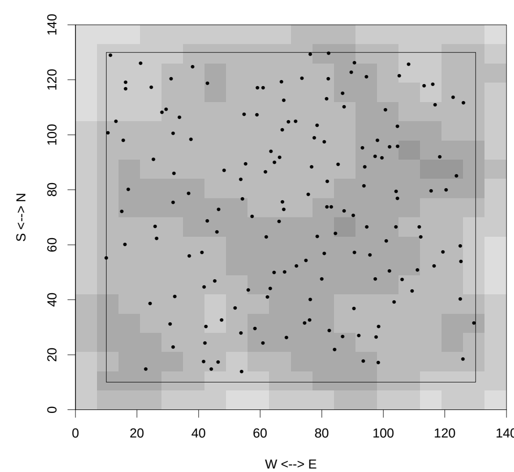


Refinement:

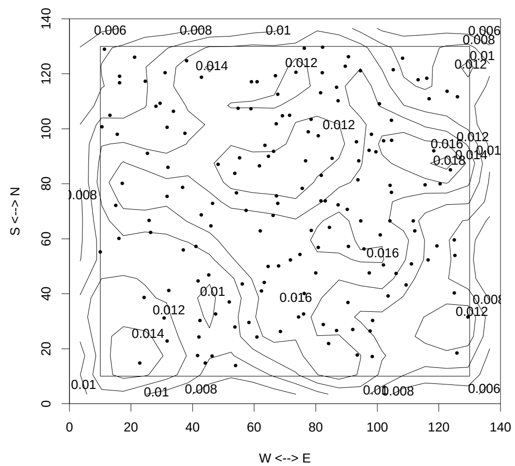
Poisson random fields are **infinitely divisible**— so we can refine from a 5×5 grid to a 100×100 grid to a 1000×1000 grid, but of course the calculations get harder. It is better to go all the way to $\infty \times \infty$— and pass to a **Poisson Random Field** $N(dx)$, and at the same time replace $\{\Gamma_j\}$ with a **Gamma Random Field** $\mathsf{Ga}(ds)$, leading to the **Continuous Poisson/Gamma Mixture Model**:

Parameter:	θ	$\sim$	$\pi(\theta)$
Impulses:	$\Gamma(ds)$	$\sim$	$\mathsf{Ga}\big(\alpha^\theta(ds), \tau^\theta(s)\big)$
Intensities:	$\Lambda(x)$	$\equiv$	$\int_{\mathcal{S}} k^\theta(x,s)\,\Gamma(ds)$
Point counts:	$N(dx)$	$\sim$	$\mathsf{Po}\big(\Lambda(s)\,m(ds)\big)$

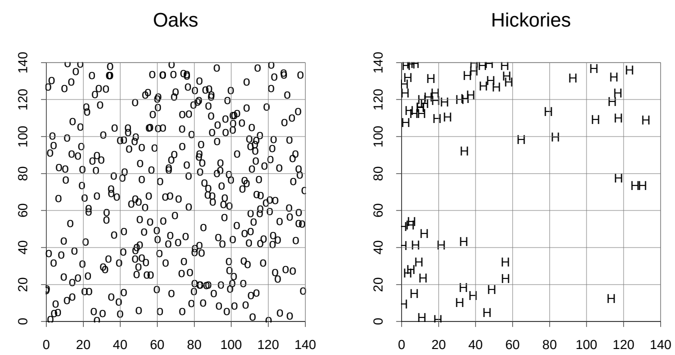
Continuous Spatial Oaks:



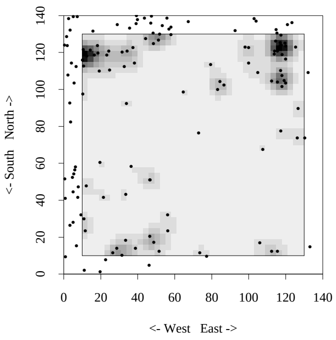
Continuous Spatial Oaks:

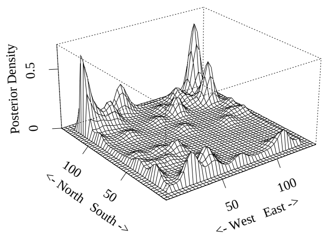


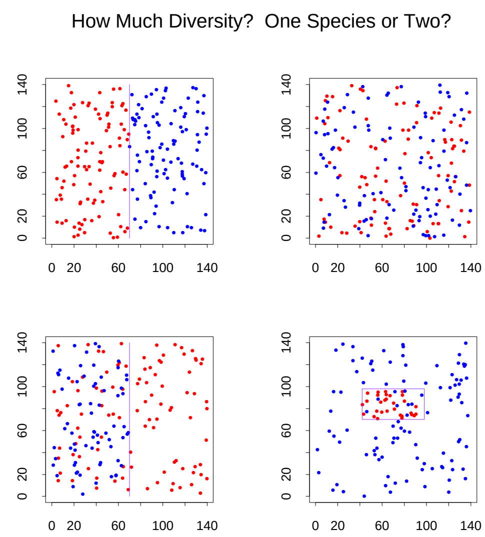
Oaks and Hickories:



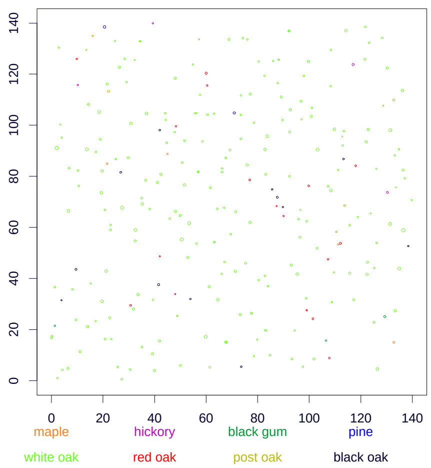
Continuous Spatial Hickories:







Spatial Biodiversity:



Biodiversity and Hill's Number

A traditional measure of the "uniformity" of a probability distribution $\{p_i\}$ is its *Shannon Entropy*

$$H(\vec{p}) \equiv -\sum p_i \log p_i$$

Always positive, this measure is bounded above by $0 \le H \le \log n$ if $p = \{p_1, ..., p_n\}$ is concentrated on n points— that limit is attained if the $\{p_i\}$ are all equal (necessarily to $1/n$).

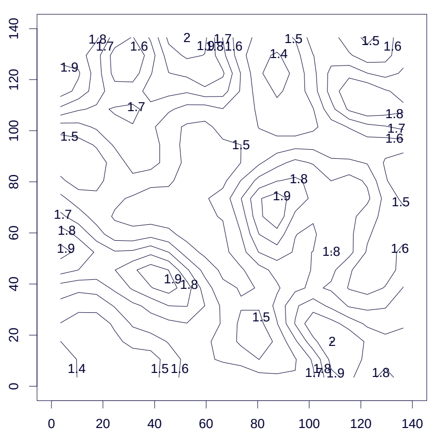
Mark Hill's (1973) index H_1:

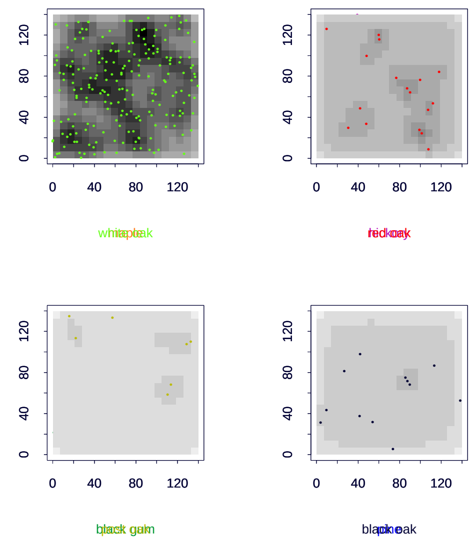
$$1 \le H_1 \equiv \exp(H) = \prod \left(\frac{1}{p_i}\right)^{p_i} \le n$$

"Equivalent Number of Species"— also the limit as $\alpha \to 1$ of

$$1 \le H_\alpha \equiv \left(\sum {p_i}^\alpha\right)^{1/(1-\alpha)} \le n$$

Spatial Hill's Number H_1:





Related papers:

- K. Ickstadt and R.L. Wolpert (1997). Multiresolution assessment of forest inhomogeneity. In *Case Studies in Bayesian Statistics, Volume III,*, C. Gatsonis *et al.*, eds., 371–386. Springer-Verlag, Berlin.
- R.L. Wolpert and K. Ickstadt (1998) Poisson/gamma random field models for spatial statistics. *Biometrika* **85**, 251–267.
- M.O. Hill (1973) Diversity and evenness: A unifying notation and its consequences. *Ecology* **54**, 427–432.
- R.L. Wolpert, K. Ickstadt, N. Christensen, R. Peet (2002) Spatial Biodiversity (*in preparation*).