Pyroclastic Flow Modeling via Lévy Processes

Simon Lunagomez and Robert Wolpert
Methodology Group Meeting

March 12, 2007

Outline

- Background
- Probability Model
- Likelihood
- How to Compute?
- Future Work

Caribbean Island of Montserrat

Goal

- Model magnitudes of pyroclastic flows as they happen on time
- Estimate probability of large eruption over $(0, T]$

The Data

- Data consists of estimated flow volumes v_j
- greater than $\epsilon = 10^4 m^3$
- J_ϵ number of events on $(0, T]$
- volumes recorded at unspecified times $\{t_j\}_{j \le J_\epsilon}$

The Data

- Several types of censoring
  1. One or several events on interval, J_ϵ known
  2. J_ϵ unknown
  3. Large event, followed by period of length d where number of events not recorded.
- We assume that the total accumulated flow is always estimated.

First Assumptions

- J_ϵ on $(0, T] \sim Po(\lambda_\epsilon T)$
- Conditioning on J_ϵ, $\{t_j\}_j \leq J_\epsilon \sim Un(0, T)$.
- and
$$f_\epsilon(v) = \alpha \epsilon^\alpha v^{-\alpha-1} \mathbf{1}_{\{v>\epsilon\}}$$
- $X_t^\epsilon \equiv \sum_{j=1}^{J_\epsilon} \{v_j \mid v_j > \epsilon, t_j > t\}$ is a compound Poisson Process.

Relation to α-Stable

- $X_t^\epsilon \equiv \sum\{v_j \mid v_j > \epsilon, t_j > t\}$ is also Lévy Process with Lévy Measure
$$\nu(dv) = \alpha\lambda v^{\alpha-1}\mathbf{1}_{\{v\}}d\nu$$
- As $\epsilon \to 0$,
$$X_t^\epsilon \Rightarrow X_t \sim St(\alpha, 1, \gamma t, \delta).$$
- here $\lambda = \lambda_\epsilon \epsilon^\alpha$.

Relation to α-Stable

- 1-1 correspondence between Lévy Processes and infinitely divisible distribution
- μ infinitively divisible if

$$\widehat{\mu}(z) = exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \zeta, z\rangle + \int_{\mathbf{R}^d}\left(e^{i\langle z,x\rangle} - 1\right)\nu(dx)\right]$$

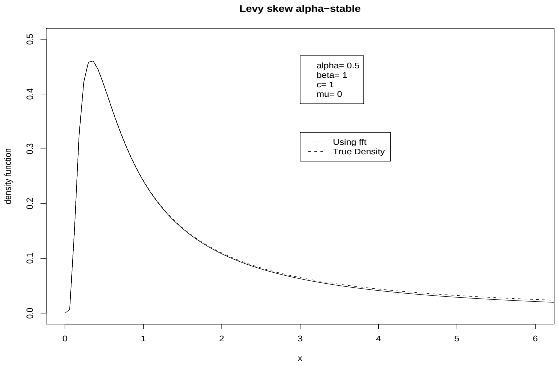
General $\mathrm{St}(\alpha, \beta, \gamma, \delta)$

- $\mathrm{St}(\alpha, \beta, \gamma, \delta)$ defined in terms of characteristic function

$$\log \phi_X(\omega) = \begin{cases} i\delta\omega - \gamma|\omega|^\alpha + i\beta\gamma \tan\frac{\pi\alpha}{2}\{|\omega|^\alpha \operatorname{sgn}\omega - \omega\} & \alpha \neq 1 \\ i\delta\omega - \gamma|\omega| - \frac{2}{\pi} i\beta\gamma\omega \log|\omega| & \alpha = 1 \end{cases}$$

$$0 < \alpha \leq 2, -1 \leq \beta \leq 1, 0 < \gamma < \infty, -\infty < \delta < \infty$$

α-Stable



Incorporating Censoring

- The Likelihood will have several components.
- Number of events is known

$$L(\alpha, \lambda) \propto \frac{(\lambda_\epsilon t)^{J_\epsilon}}{J_\epsilon!} e^{-\lambda_\epsilon t} \int_{\mathbf{S}(J_\epsilon, \epsilon, v)} \prod_{m=1}^{J_\epsilon - 1} \alpha \epsilon^\alpha v_j^{-\alpha-1} dv_j$$

Incorporating Censoring

- Number of events is unknown: Use $St(\alpha, 1, \gamma t, \delta)$.
- Here

$$\gamma = \frac{\pi\lambda}{\{2\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})\}}$$
$$\delta = \gamma \tan\left(\frac{\pi\alpha}{2}\right)$$

Incorporating Censoring

- Large event followed by absence of recordings

$$L(\alpha, \lambda) \propto \lambda_\eta e^{-\lambda_\eta t} \int_\eta^{v-\eta} f(y \mid \alpha, \eta) g(v - y \mid \alpha, \lambda) dy$$

- where $f \sim Pa(\alpha, \eta)$, and $g \sim St(\alpha, 1, \gamma d, \delta)$

Ideas

- Use importance sampling

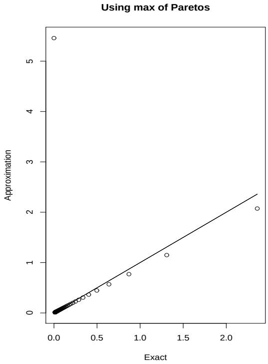
$$\int \frac{f(v\mathbf{y} \mid \alpha)}{h(\mathbf{w} \mid c)} h(\mathbf{w} \mid c) d\mathbf{w}$$

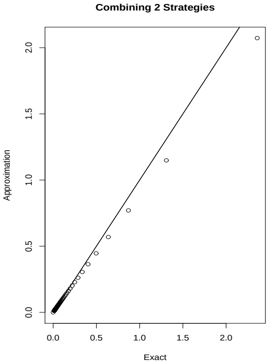
- where $h(\mathbf{w} \mid c)$ is $Dirich\left(\frac{c}{J_\epsilon}, \ldots, \frac{c}{J_\epsilon}\right)$.

Ideas

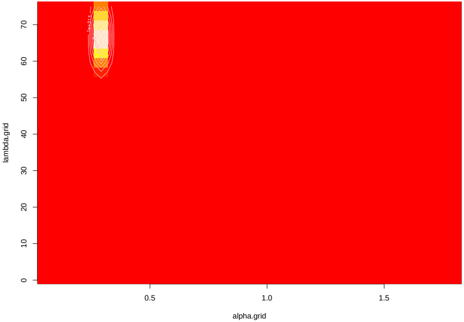
- Sum of independent $Pa(\alpha, \eta)$ can be approximated by the maximum.

Testing Approximations





(Wrong!) Likelihood Region



- Obtain likelihood regions using revised database
- Fit Bayesian model to perform predictions

References

- Pittman, E. B. (2006) Granular avalanche modeling. Overheads from 2006-10-02 presentation in SAMSI Granular working group.
- Robert, C.P., Casella, G. (2005) Monte Carlo Statistical Methods, Second Edition. Springer-Verlag, New York.
- Sato, K.I. (1999) Lévy Processes and Infinitely Divisible Distributions . Cambridge
- Wolpert, R.L. (2006) A Simple Pyroclastic Flow Model. Notes for SAMSI Granular working group.