

dukeblue **Quantifying Elephant Social Structure:  
Using a Bilinear Mixed Effects Model to Elicit  
Qualities of Elephant Behavior**

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## dukeblue Elephant Social Structure



- Only females and juveniles form families. Adult males just run around looking to mate.
- The oldest female tends to be the leader since she's the largest and wisest.
- Elephants within a family are often related.

## dukeblue Scientific Questions



- What factors influence the development and evolution of social structure in elephants?
- What role does kinship play in the social structure?
- How does the social structure change in the Wet Season vs. the Dry Season?

## dukeblue Data Collection



- Biologists in Kenya ride into the National Park looking for herds of elephants.
- The adult female elephants are identified and recorded.
- The biologists either stay to observe the family or move on to find another herd.

## dukeblue Binomial Data Example

The total number of times data were recorded:

$$\text{Dry Season: } N_{\text{Dry}} = 418$$

$$\text{Wet Season: } N_{\text{Wet}} = 219$$

If Amy, Ang, and Aud are observed together, and the others are missing, then:

$$y_{\text{AmyAng}} = y_{\text{AmyAud}} = y_{\text{AngAud}} = 1$$

$$n_{\text{AmyAng}} = n_{\text{AmyAud}} = n_{\text{AngAud}} = 1$$

While for one missing elephant:

$$y_{\text{AmyAga}} = 0$$

$$n_{\text{AmyAga}} = 1$$

Whereas for two missing elephants:

$$y_{\text{AliAga}} = 0$$

$$n_{\text{AliAga}} = 0$$

dukeblue **Binomial Data, Cont.**

In this example of five elephants Amy, Angelina, Audrey, Alison, and Agatha at time =  $t$ , the  $\mathbf{y}$  matrix of successful observations would be:

$$\mathbf{y}_t = \begin{array}{c|ccccc} & \text{Amy} & \text{Ang} & \text{Aud} & \text{Ali} & \text{Aga} \\ \hline \text{Amy} & \ddots & 1 & 1 & 0 & 0 \\ \text{Ang} & 1 & \ddots & 1 & 0 & 0 \\ \text{Aud} & 1 & 1 & \ddots & 0 & 0 \\ \text{Ali} & 0 & 0 & 0 & \ddots & \mathbf{0} \\ \text{Aga} & 0 & 0 & 0 & \mathbf{0} & \ddots \end{array}$$

The  $\mathbf{n}_t$  matrix of potential observations =

	Amy	Ang	Aud	Ali	Aga
Amy	$\ddots$	1	1	1	1
Ang	1	$\ddots$	1	1	1
Aud	1	1	$\ddots$	1	1
Ali	1	1	1	$\ddots$	<b>0</b>
Aga	1	1	1	<b>0</b>	$\ddots$

## dukeblue The Model

- Data are binomial observations on pairs of elephants
  - $y_{ij} \sim \text{Bin}(n_{ij}, p_{ij})$
  - $y_{ij}$  is the number of times elephants  $i$  and  $j$  observed together.
  - $n_{ij}$  is the number of times either  $i$  or  $j$  observed.
- Use a Generalized Linear Model  $\implies$  Logistic regression
  - $E(y_{ij} | \mathbf{red}\theta_{ij}) = g(\mathbf{red}\theta_{ij})$ .
  - $g$  is the inverse logit link function.
  - The probability of elephants  $i$  and  $j$  being together is:

$$p_{ij} = \frac{\exp \mathbf{red}\theta_{ij}}{1 + \exp \mathbf{red}\theta_{ij}}.$$

- $\mathbf{red}\theta_{ij}$  **is the linear predictor.**

## dukeblue **Linear Predictor** $\theta_{ij}$

### red **How often are two elephants together?**

- Common intercept **red**  $\beta_0$ , a baseline probability.
- Intrinsic sociability **red**  $a_i$  random effect.
  - Sociable elephants will more often be observed in large groups.
- Kinship relatedness **red**  $\beta_k k_{ij}$ .
  - 1- DNA relatedness measure  $k_{1ij}$ : how closely elephants  $i$  and  $j$  are related
  - 2- Mother/Daughter pair indicator  $k_{2ij}$
  - 3- Sisters pair indicator  $k_{3ij}$
- Normal error **red**  $\gamma_{ij}$  (unexplained error or white noise).
- **dukeblue** Pairwise effect  $z'_i z_j$  between elephants  $i$  and  $j$ .

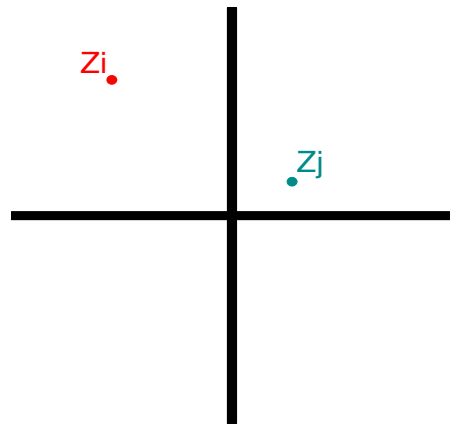
$$\text{red}\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \text{dukeblue} z'_i z_j$$

## dukeblue Pairwise Effects

$\mathbf{z}_i' \mathbf{z}_j$  is the inner product of the positions of elephants  $i$  and  $j$  in latent (unobserved) Social Space.

- I choose the dimension of social space  $k = 2$ .

Elephants  $i$  and  $j$  have positions  $\mathbf{z}_i$  and  $\mathbf{z}_j$  in 2D social space.



$$\mathbf{z}_i \sim N(\mathbf{0}, \sigma_z^2 \mathbf{I})$$

$$\mathbf{z}_j \sim N(\mathbf{0}, \sigma_z^2 \mathbf{I})$$

- If  $\mathbf{z}_i' \mathbf{z}_j = 0$  then elephants  $i$  and  $j$  interact as often as their sociabilities  $a_i, a_j$  and their kinships  $k_{ij}$  would predict.
- If  $\mathbf{z}_i' \mathbf{z}_j > 0$  then  $i$  and  $j$  like each other and are observed together more often than the model would otherwise predict.
- If  $\mathbf{z}_i' \mathbf{z}_j < 0$  then  $i$  and  $j$  dislike each other.



- The inner product of two vectors is the similarity of their directions, scaled by their lengths.

## dukeblue Vague Priors

$$\text{red}\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

Intercept:  $\beta_0 \sim N(0, 100)$

Sociabilities:  $a_i, a_j \sim N(0, \sigma_{\text{soc}}^2), \quad \sigma_{\text{soc}}^2 \sim IG(\frac{1}{2}, \frac{1}{2})$

Kinship Coefficients:  $\beta_k \sim N(\mathbf{0}, \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix})$

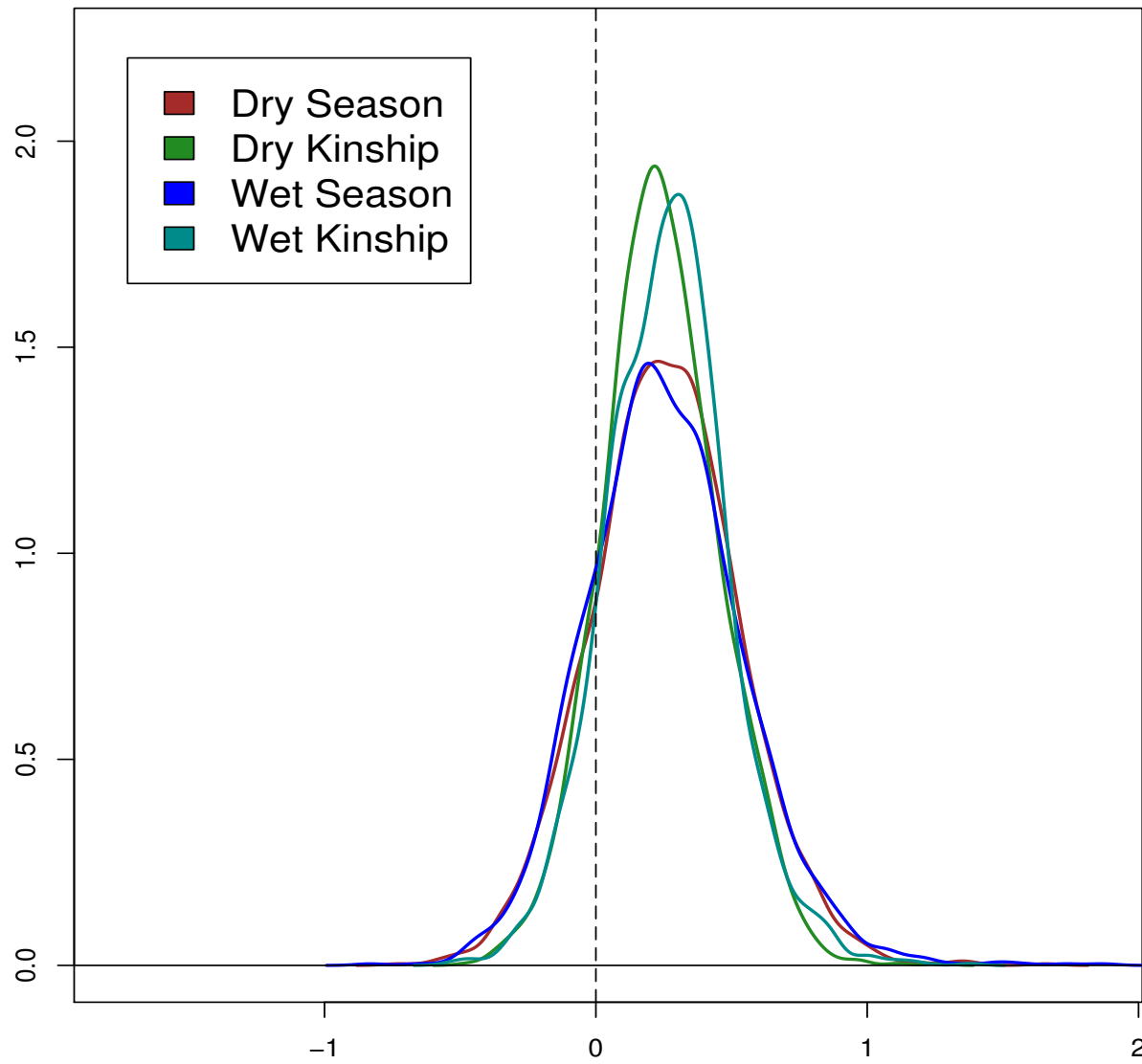
Pairwise error:  $\gamma_{ij} \sim N(0, \sigma_\gamma^2), \quad \sigma_\gamma^2 \sim IG(\frac{1}{2}, \frac{1}{2})$

Social space:  $\mathbf{z}_i, \mathbf{z}_j \sim N(\mathbf{0}, \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix}), \quad \sigma_z^2 \sim IG(\frac{1}{2}, \frac{1}{2})$

dukeblue **Results for Amy, Matriarch of Family AA**



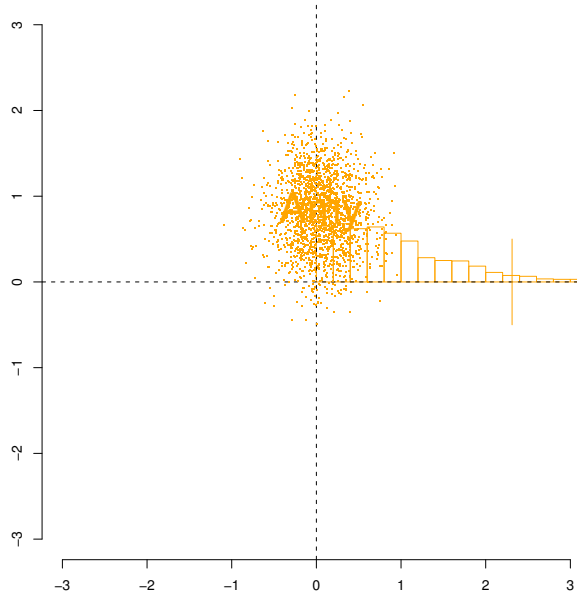
dukeblue Amy Sociability Posterior Density  $\text{reda}_{\text{Amy}}$



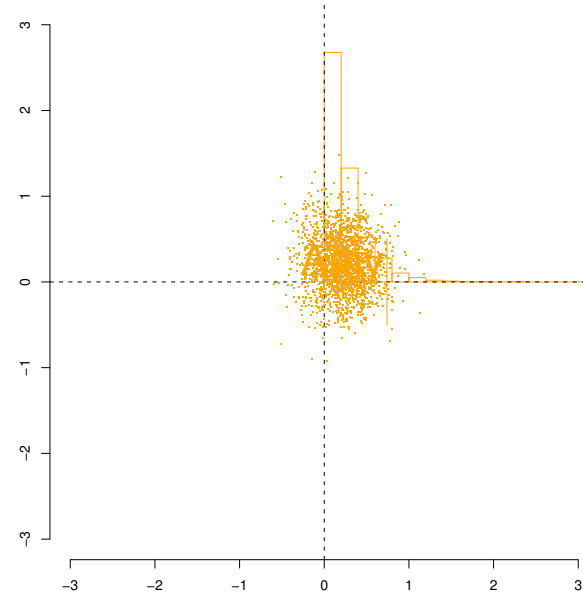
$$\theta_{ij} = \beta_0 + \text{reda}_i + \mathbf{a}_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

# dukeblue Amy's Posterior Pickiness

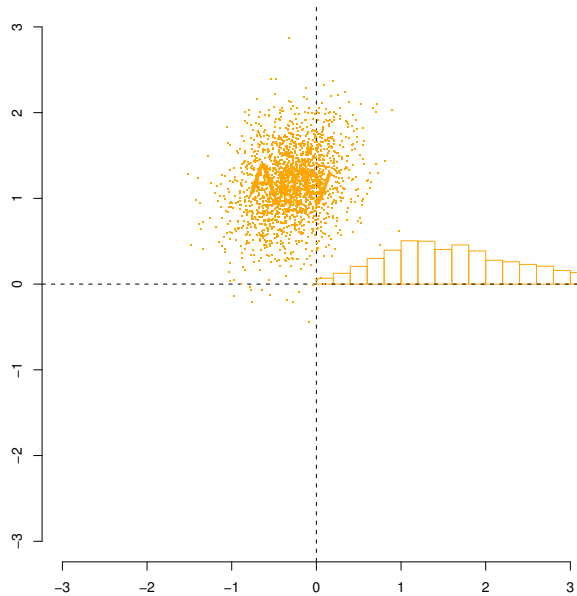
Amy Pickiness Dry Season



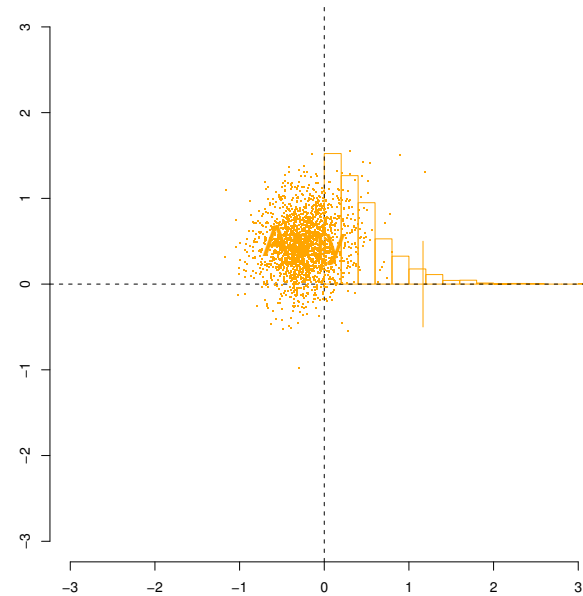
Amy Pickiness Dry Kinship



Amy Pickiness Wet Season



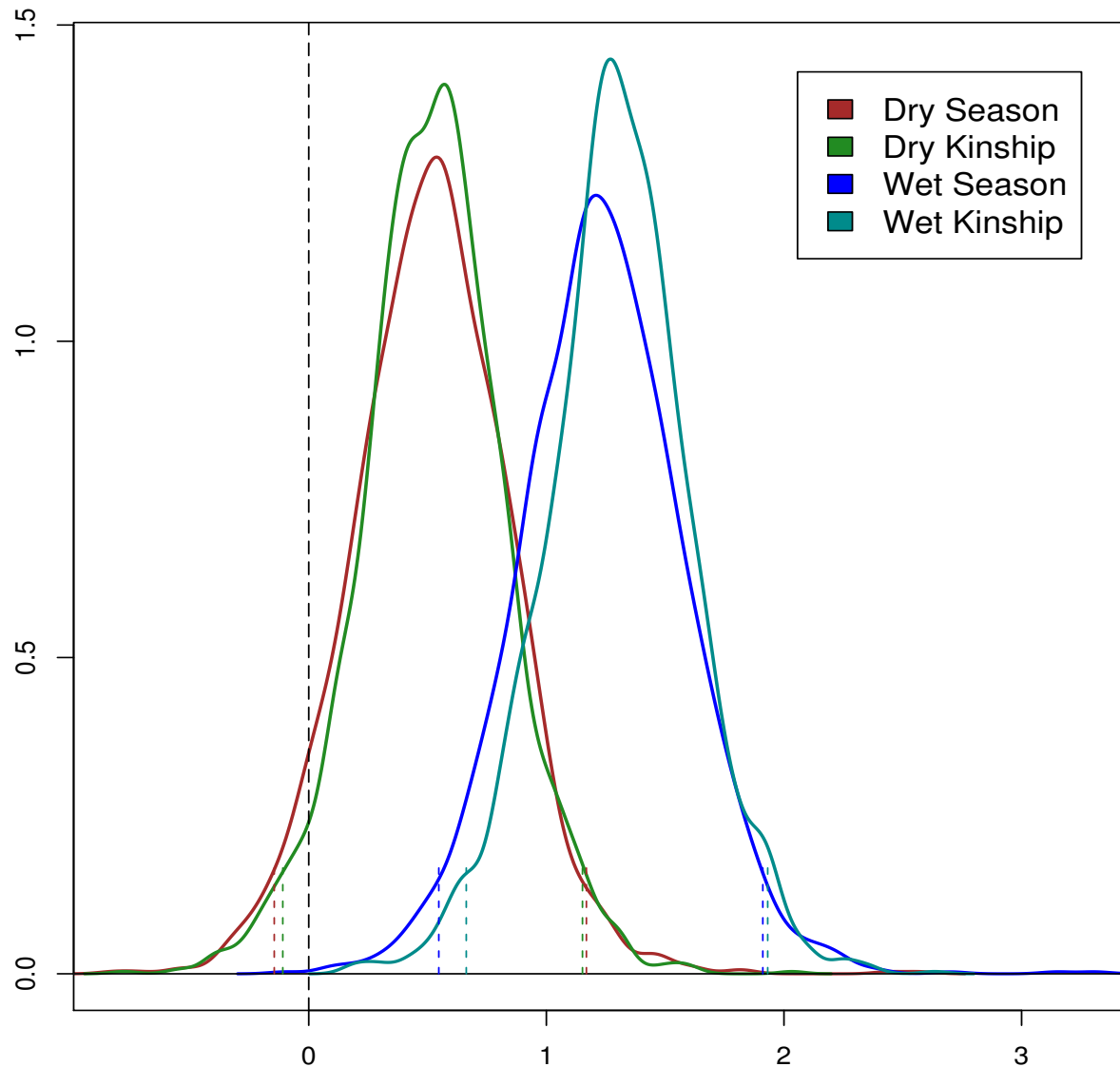
Amy Pickiness Wet Kinship



dukeblue **Elephant Family Results**

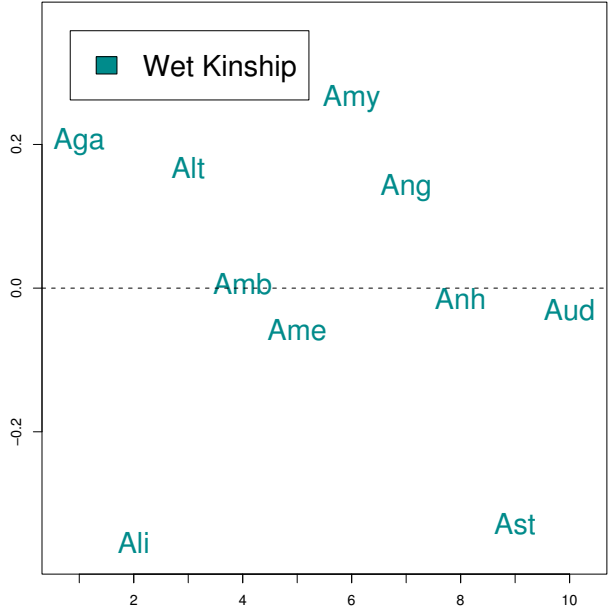
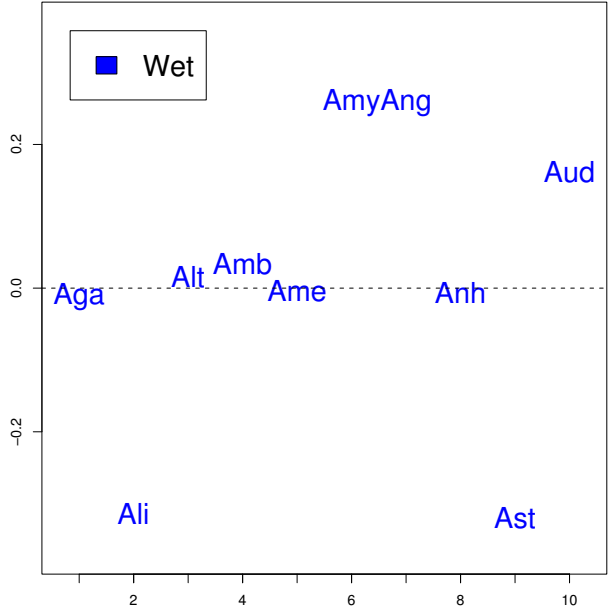
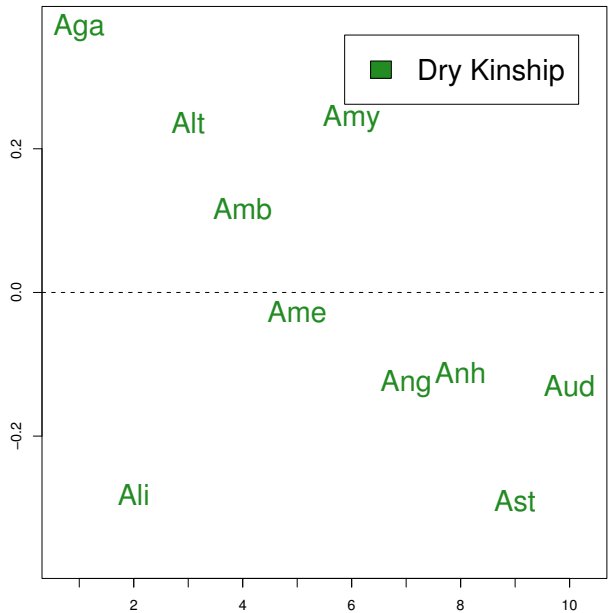
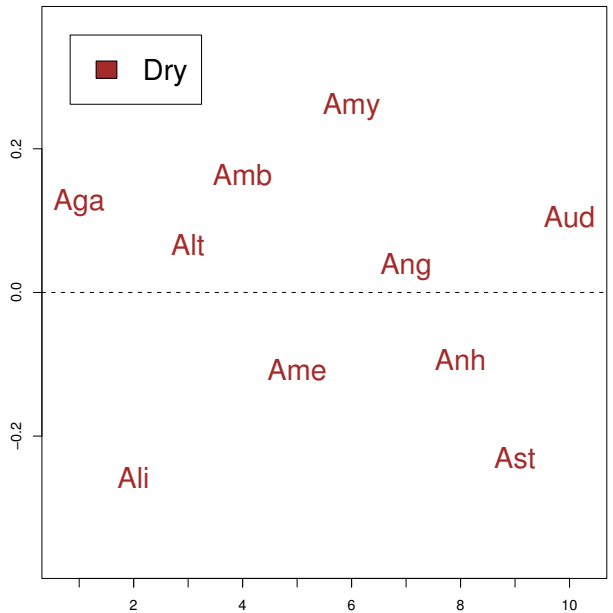


dukeblue **Posterior Intercepts** red  $\beta_0$



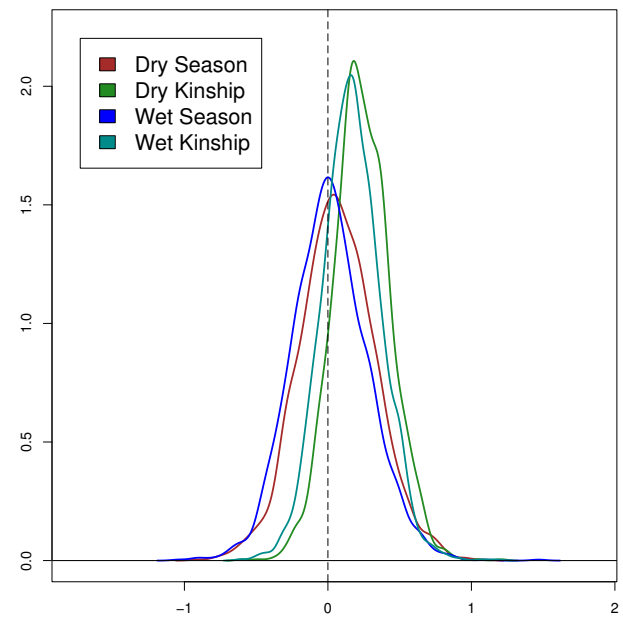
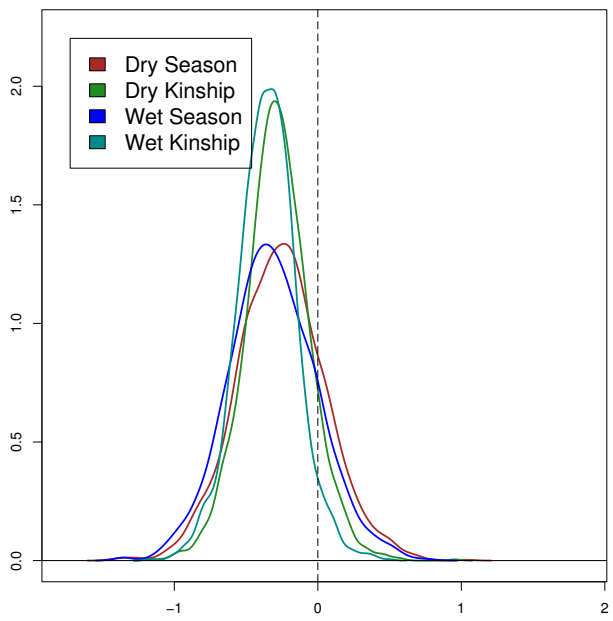
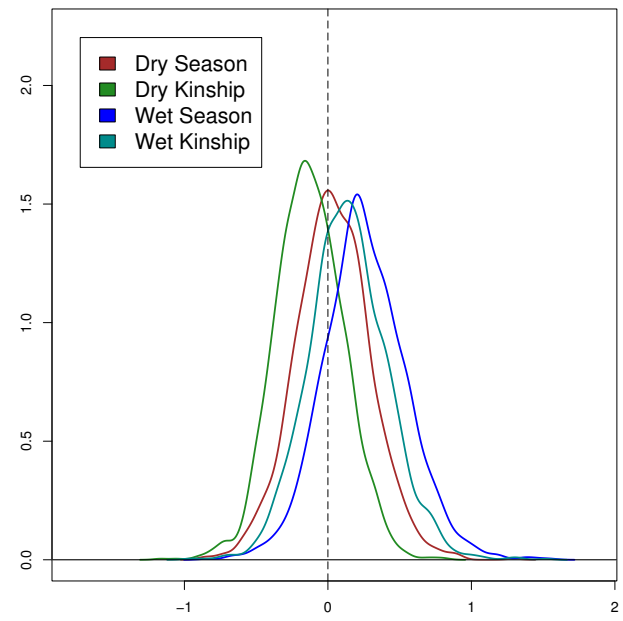
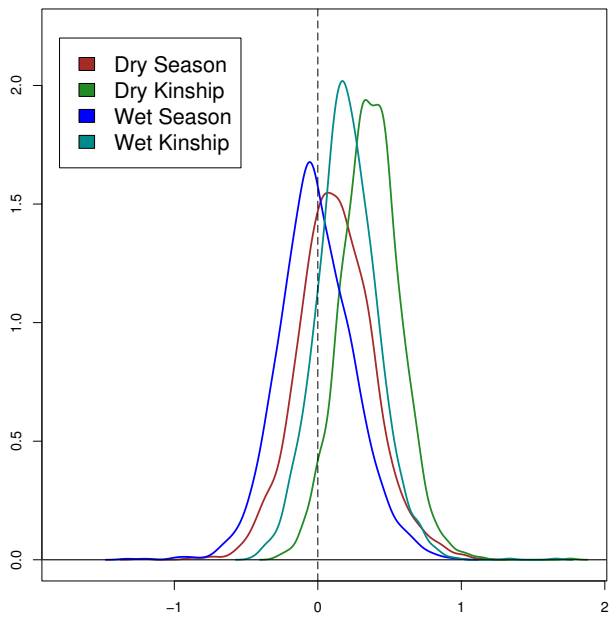
$$\theta_{ij} = \text{red}\beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

dukeblue **Posterior Sociabilities** red  $\bar{a}$

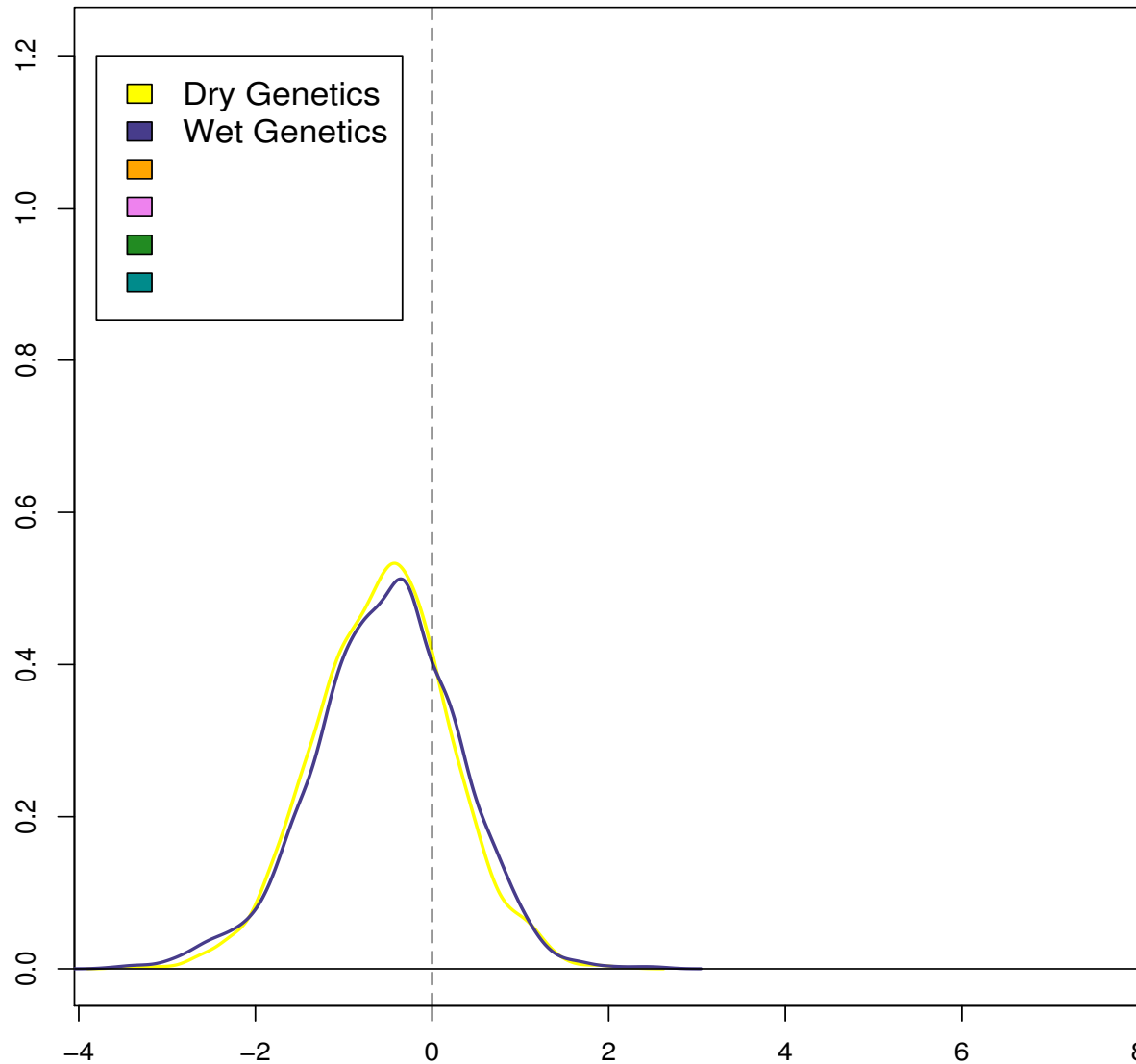


$$\theta_{ij} = \beta_0 + \text{red}a_i + \text{red}a_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

# dukeblue Sociabilities for Aga, Ang, Ali, Alt

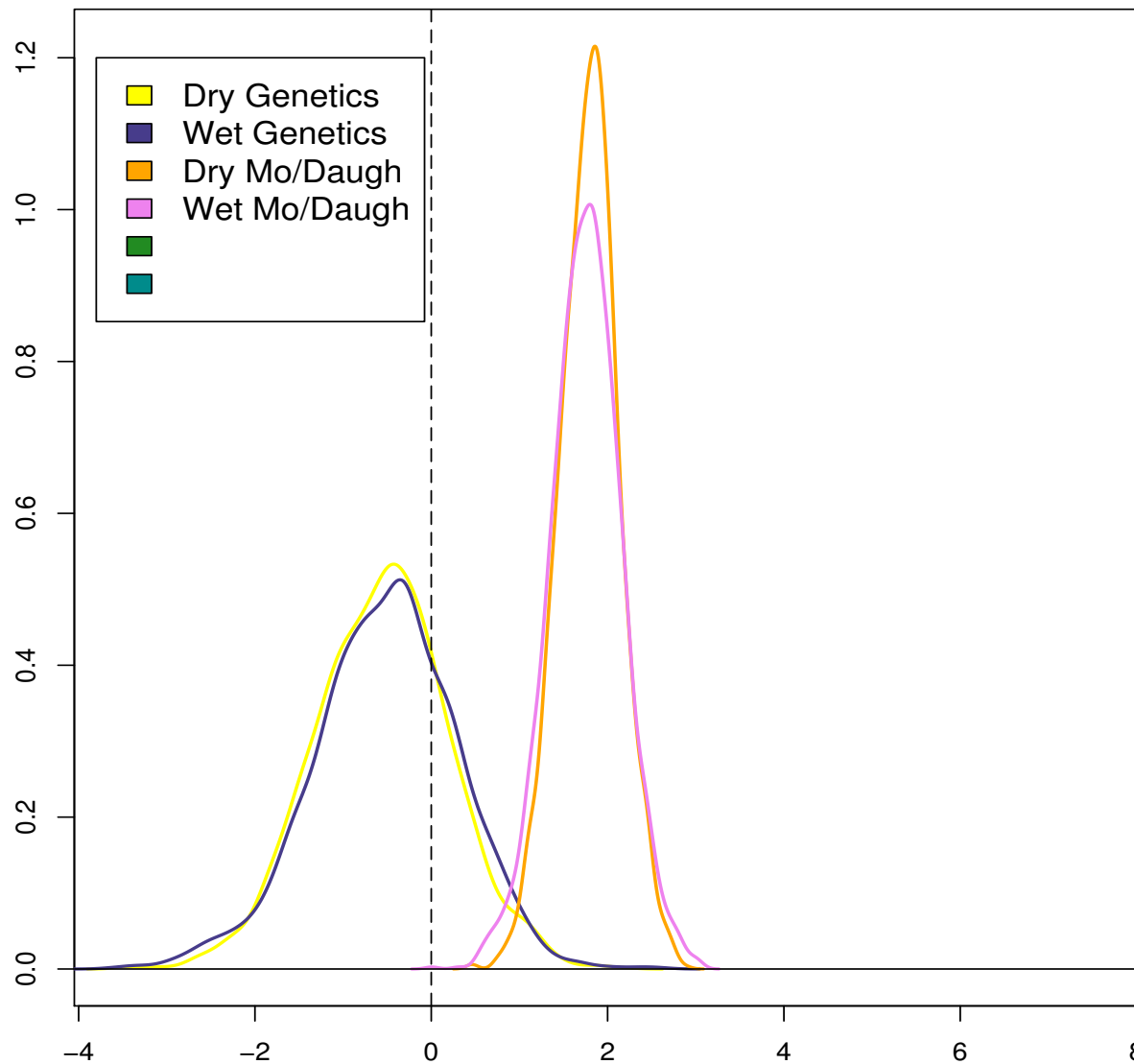


# dukebluePosteriors for Kinship Coefficients red $\beta_k$



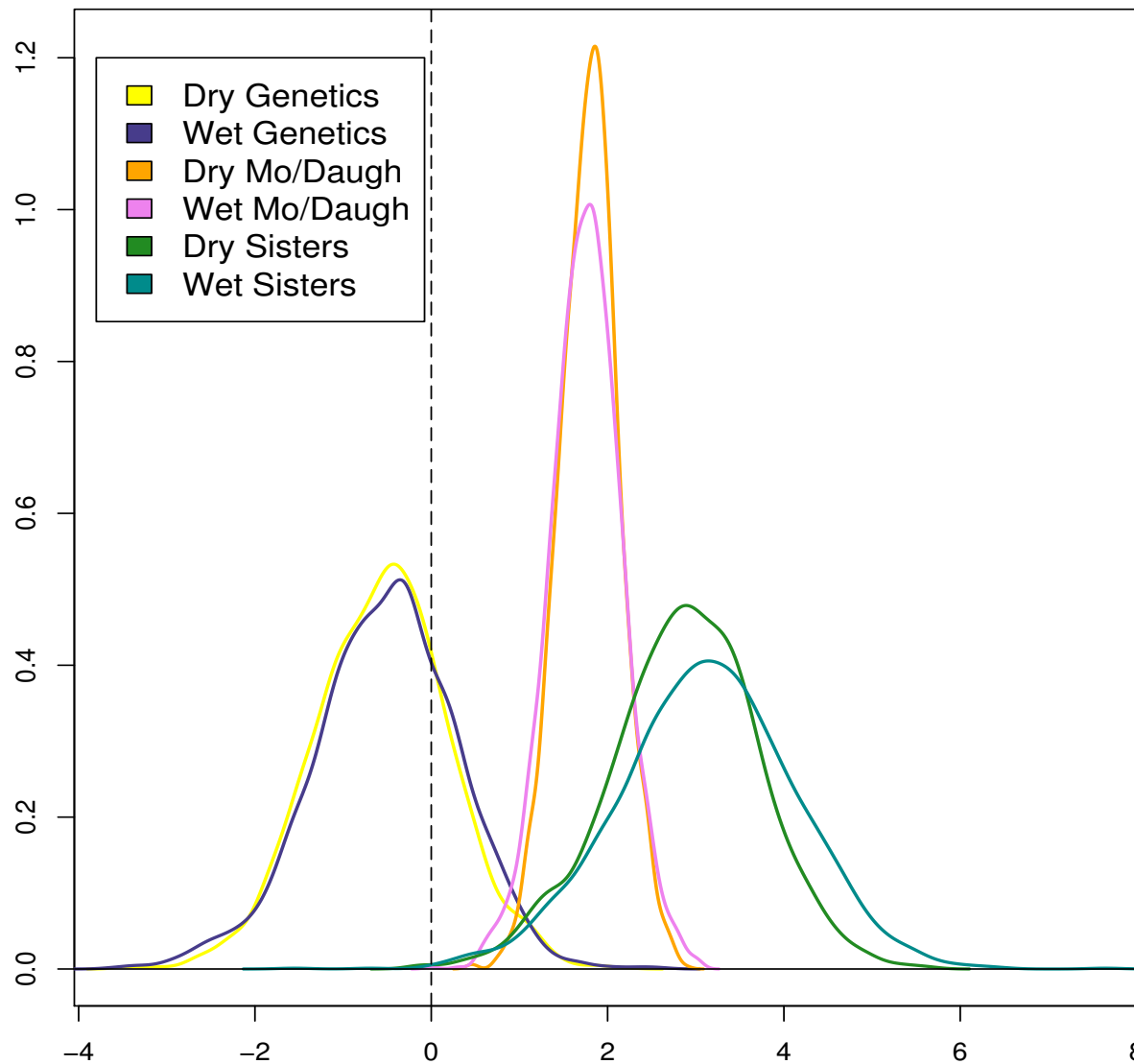
$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \text{red}\beta_{k1} \mathbf{k}_{1ij} + \beta_{k2} \mathbf{k}_{2ij} + \beta_{k3} \mathbf{k}_{3ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

# dukebluePosteriors for Kinship Coefficients $\text{red}\beta_k$



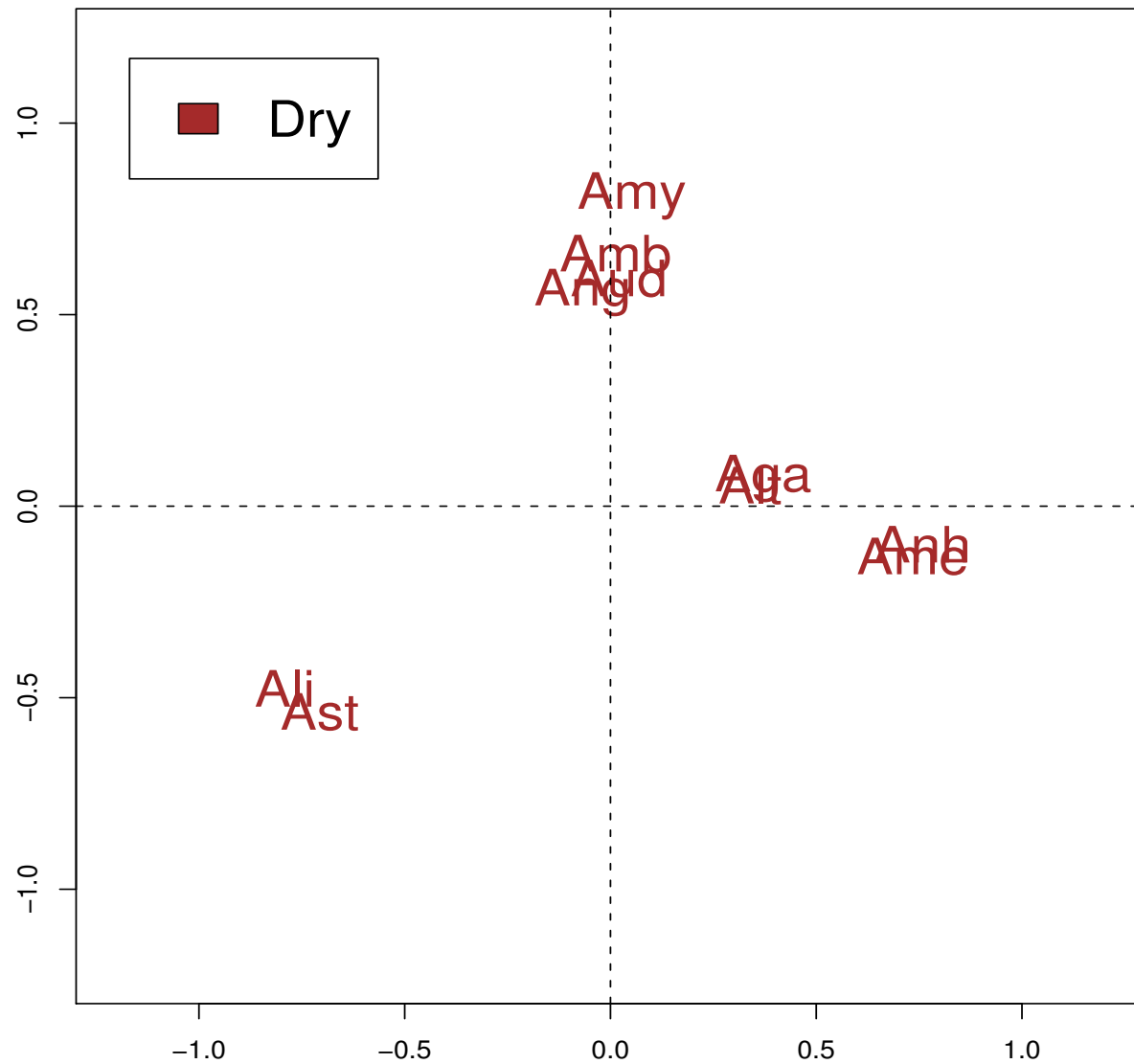
$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \text{red}\beta_{k1} \mathbf{k}_{1ij} + \text{red}\beta_{k2} \mathbf{k}_{2ij} + \beta_{k3} \mathbf{k}_{3ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

# dukeblue Posteriors for Kinship Coefficients $\text{red}\beta_k$



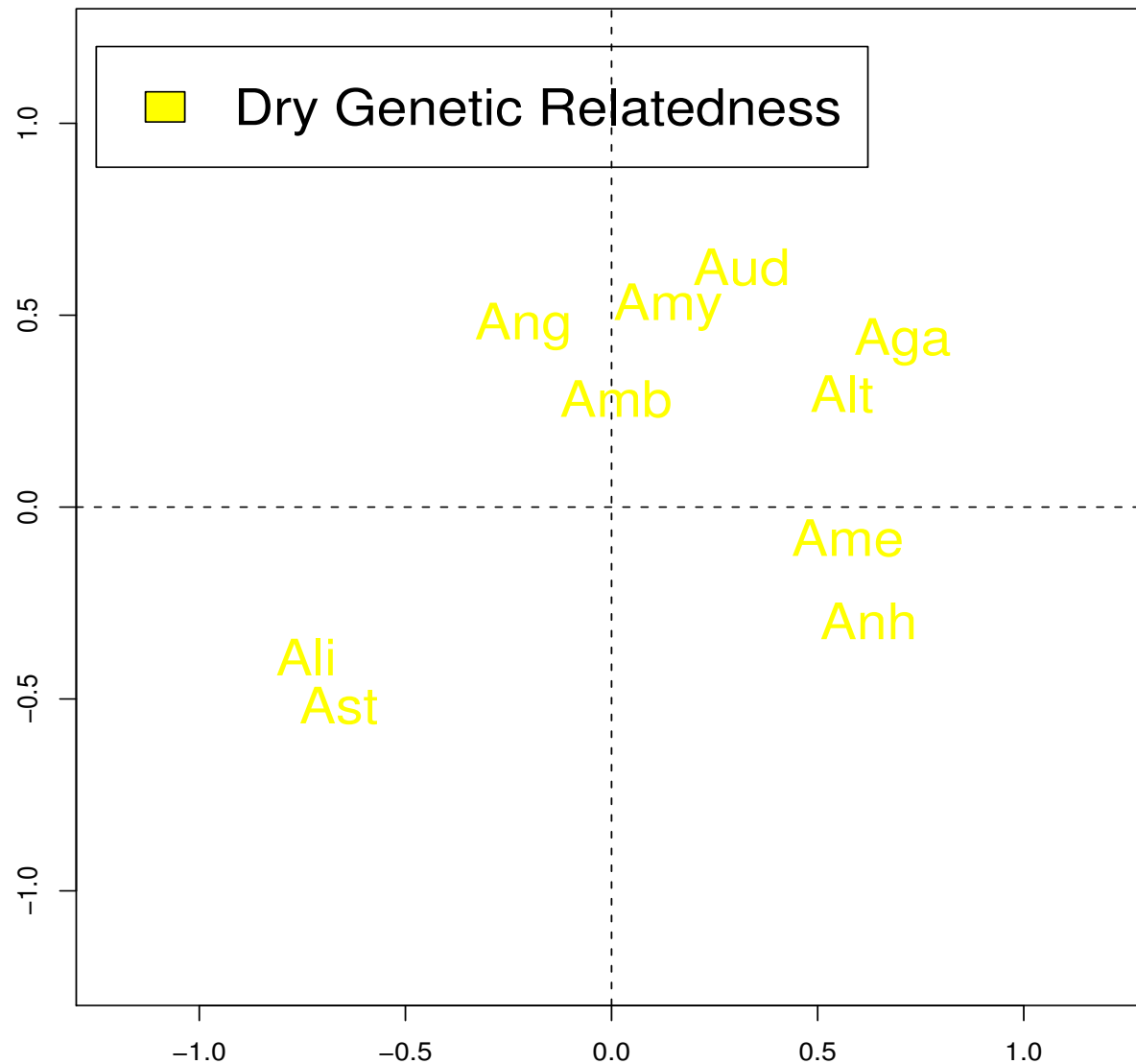
$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \text{red}\beta_{k1} \mathbf{k}_{1ij} + \text{red}\beta_{k2} \mathbf{k}_{2ij} + \text{red}\beta_{k3} \mathbf{k}_{3ij} + \gamma_{ij} + \mathbf{z}'_i \mathbf{z}_j$$

dukeblue Dry Season Social Space red  $\bar{z}_i$  Posteriors

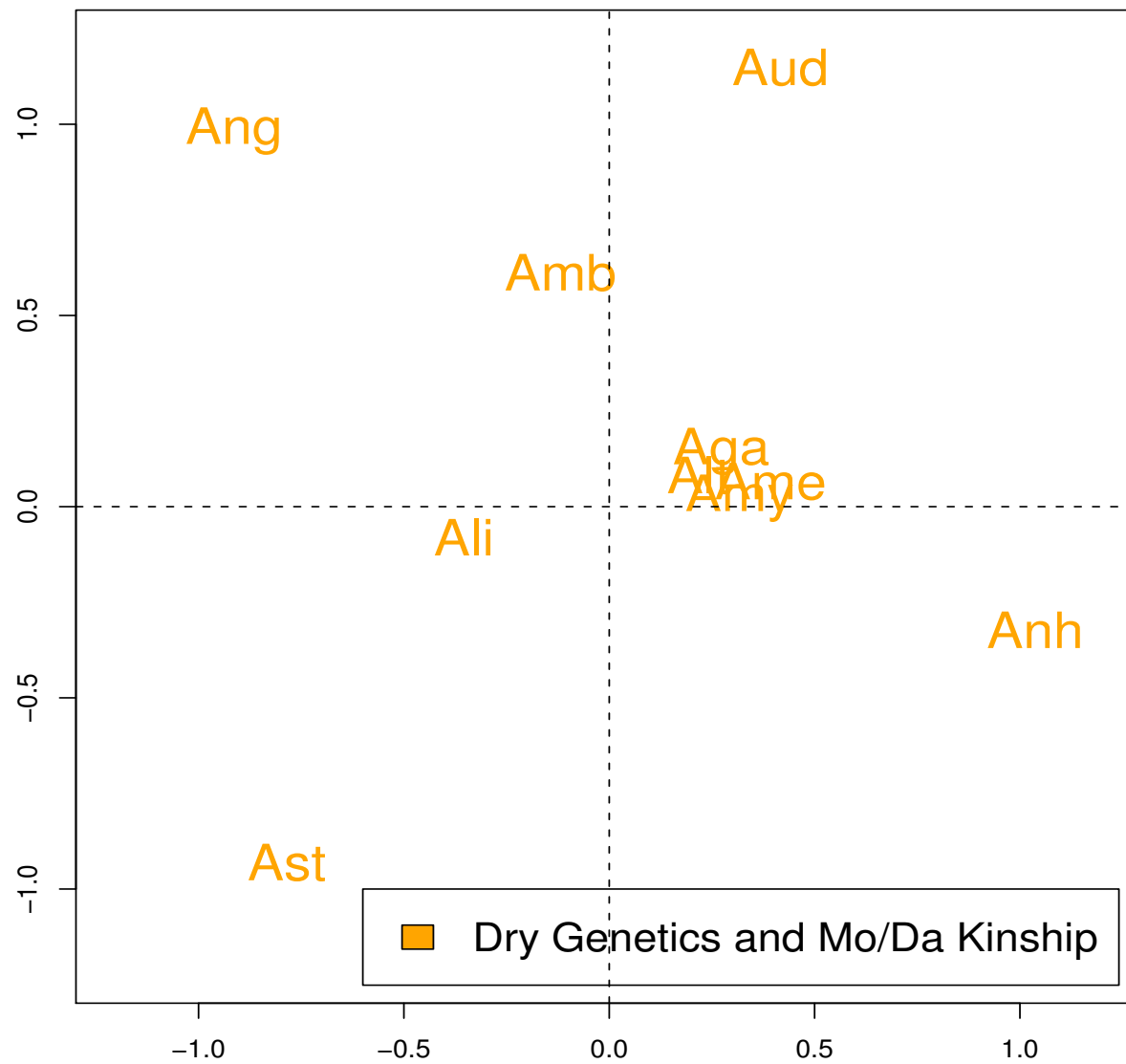


$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \gamma_{ij} + \text{red}z'_i z_j$$

dukeblue Dry Genetic DNA Social Space red $\bar{z}_i$  Posteriors

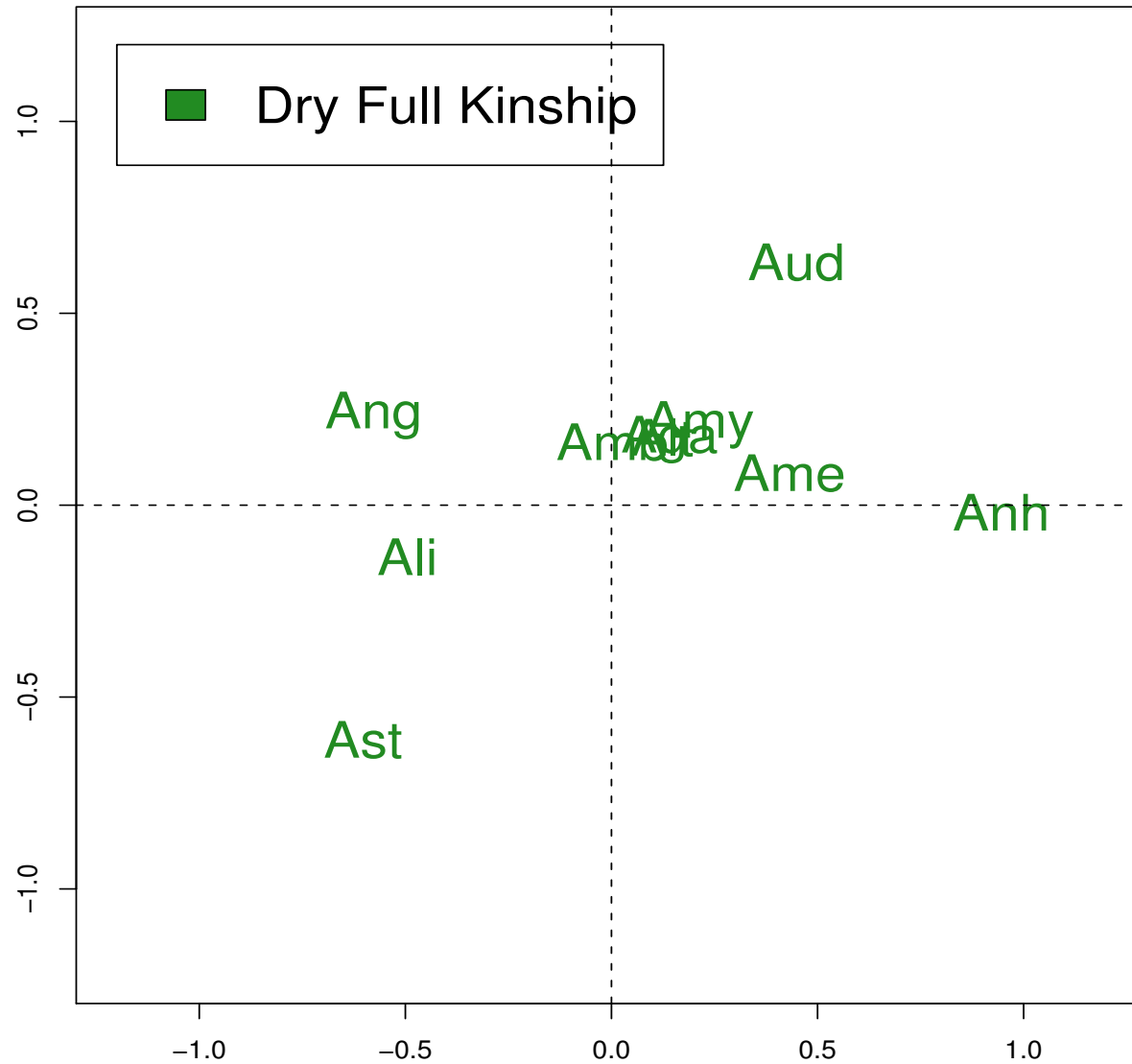


$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_{k1} \mathbf{k}_{1ij} + \gamma_{ij} + \mathbf{redz}'_i \mathbf{z}_j$$



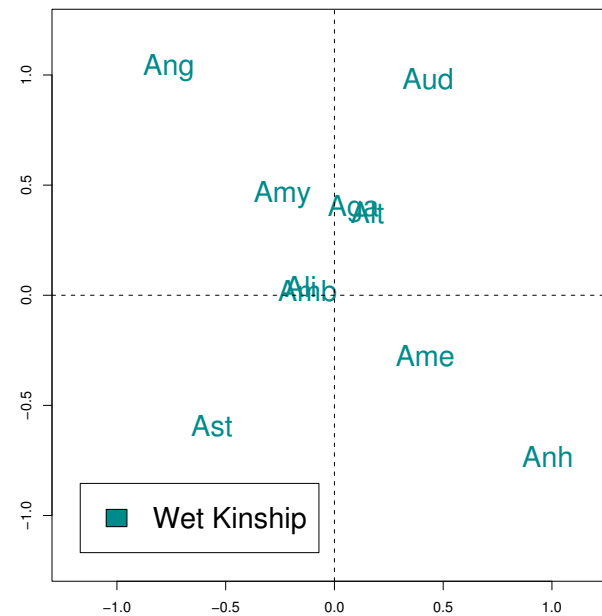
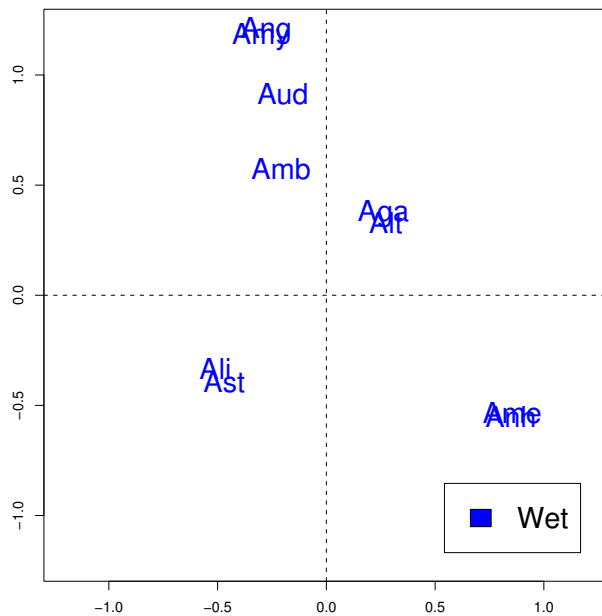
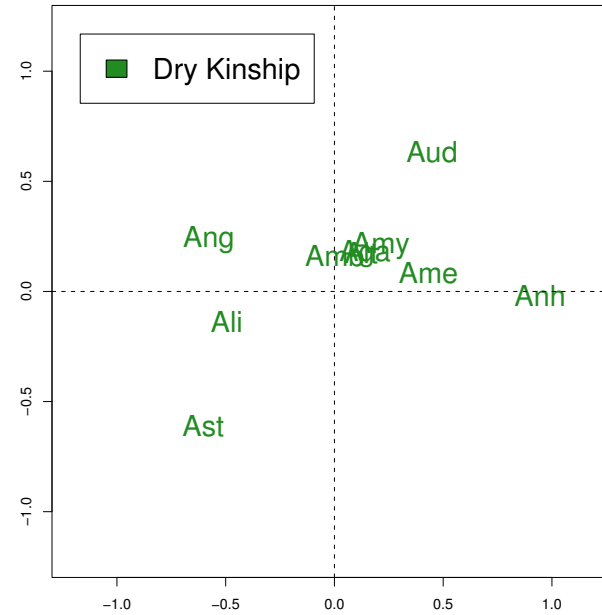
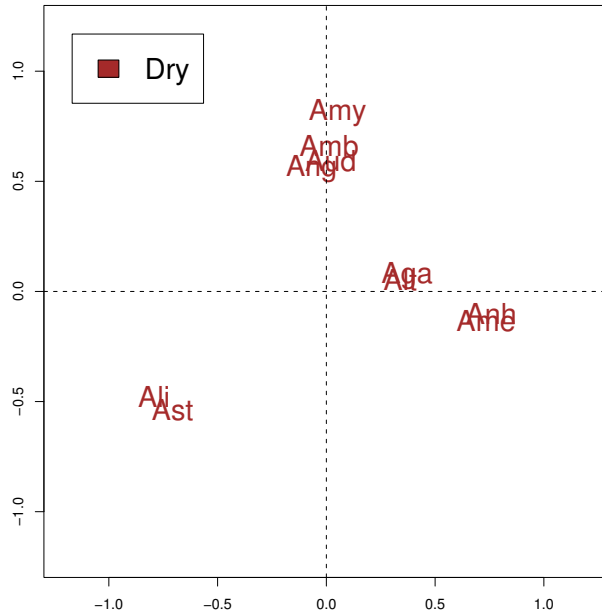
$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_{k1} \mathbf{k}_{1ij} + \beta_{k2} \mathbf{k}_{2ij} + \gamma_{ij} + \text{red}\mathbf{z}'_i \mathbf{z}_j$$

dukeblue Dry Full Kinship Social Space red $\bar{z}_i$  Posteriors

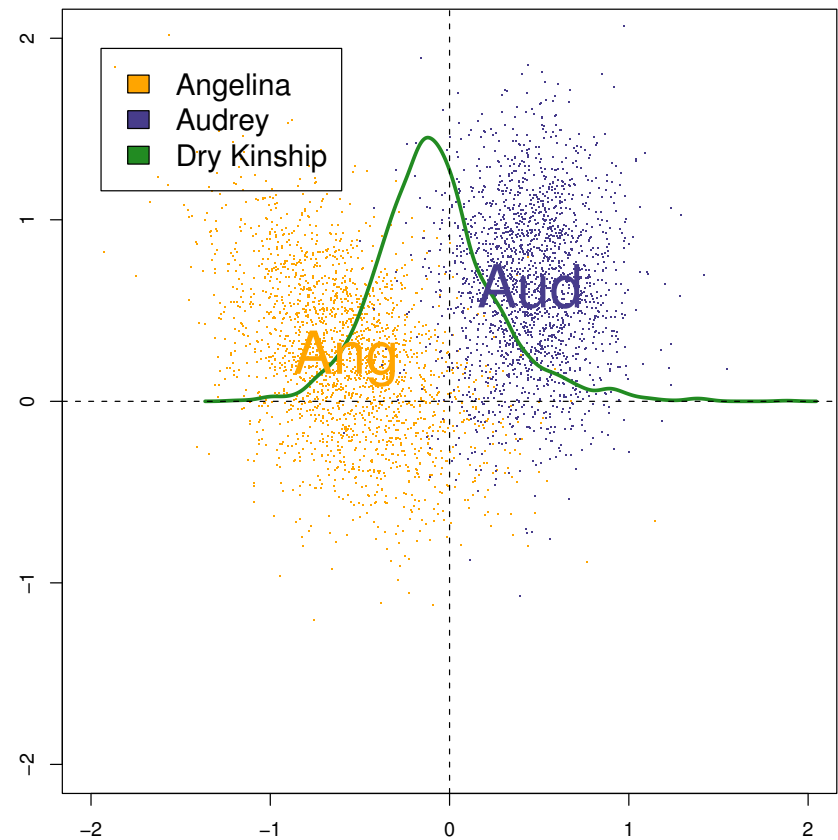
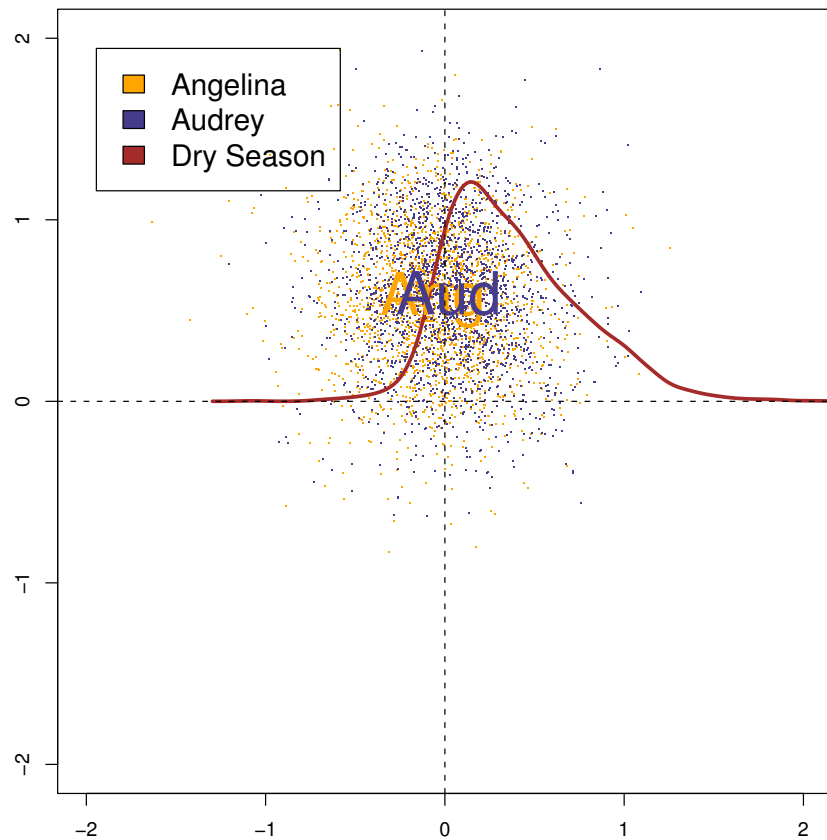


$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_{k1} \mathbf{k}_{1ij} + \beta_{k2} \mathbf{k}_{2ij} + \beta_{k3} \mathbf{k}_{3ij} + \gamma_{ij} + \mathbf{redz}'_i \mathbf{z}_j$$

# dukeblue Social Space Posterior Means $\bar{z}_i$



# dukeblue Angelina and Audrey: Two Sisters' Inner Products



$$\theta_{ij} = \beta_0 + \mathbf{a}_i + \mathbf{a}_j + \beta_k \mathbf{k}_{ij} + \gamma_{ij} + \text{red}z'_i z_j$$

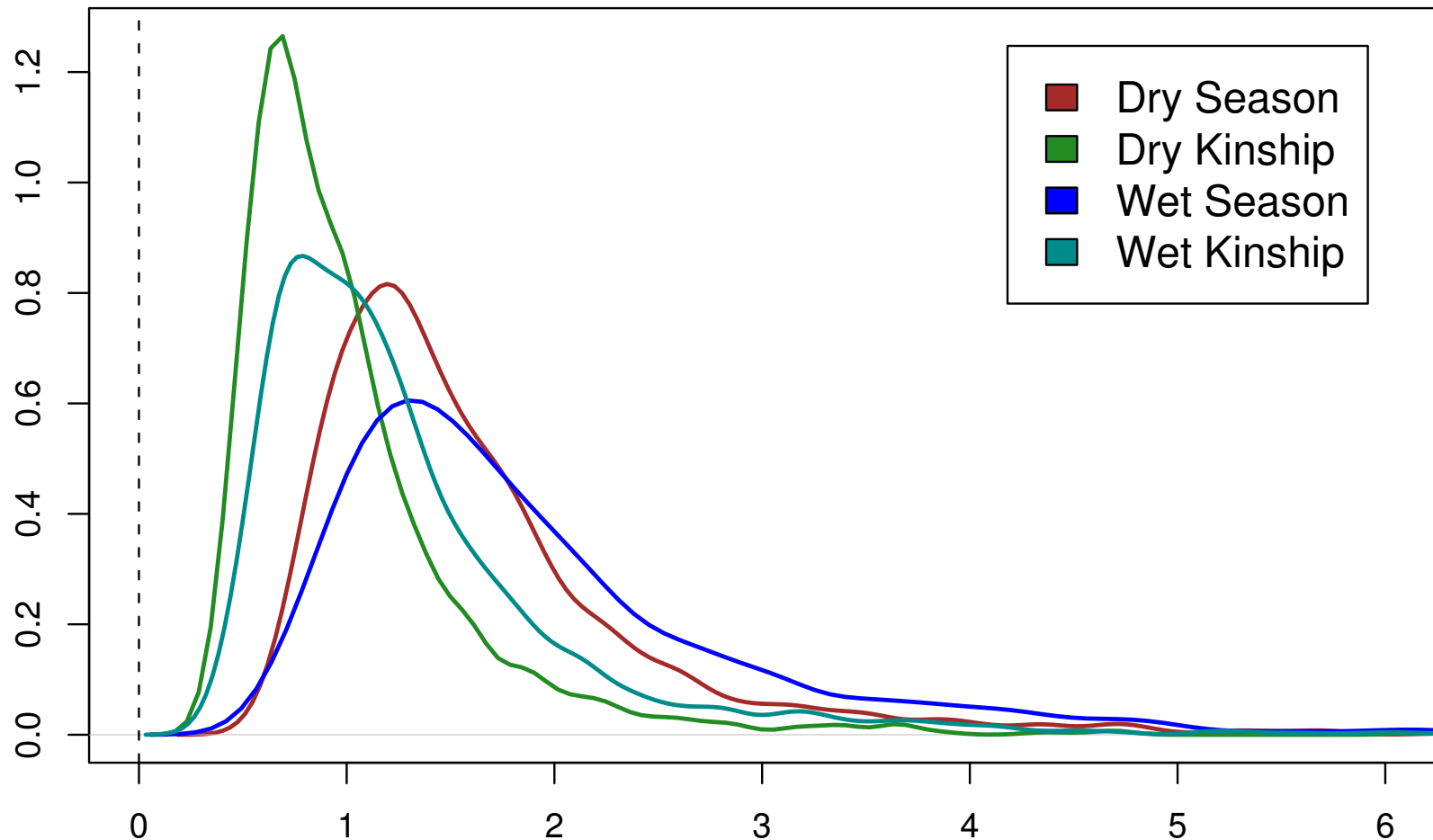
dukeblue **Relative Size of the Variance Terms in the Error  $\epsilon_{ij}$**

$$\theta_{ij} = \beta_0 + a_i + a_j + \beta_k k_{ij} + \gamma_{ij} + z'_i z_j$$

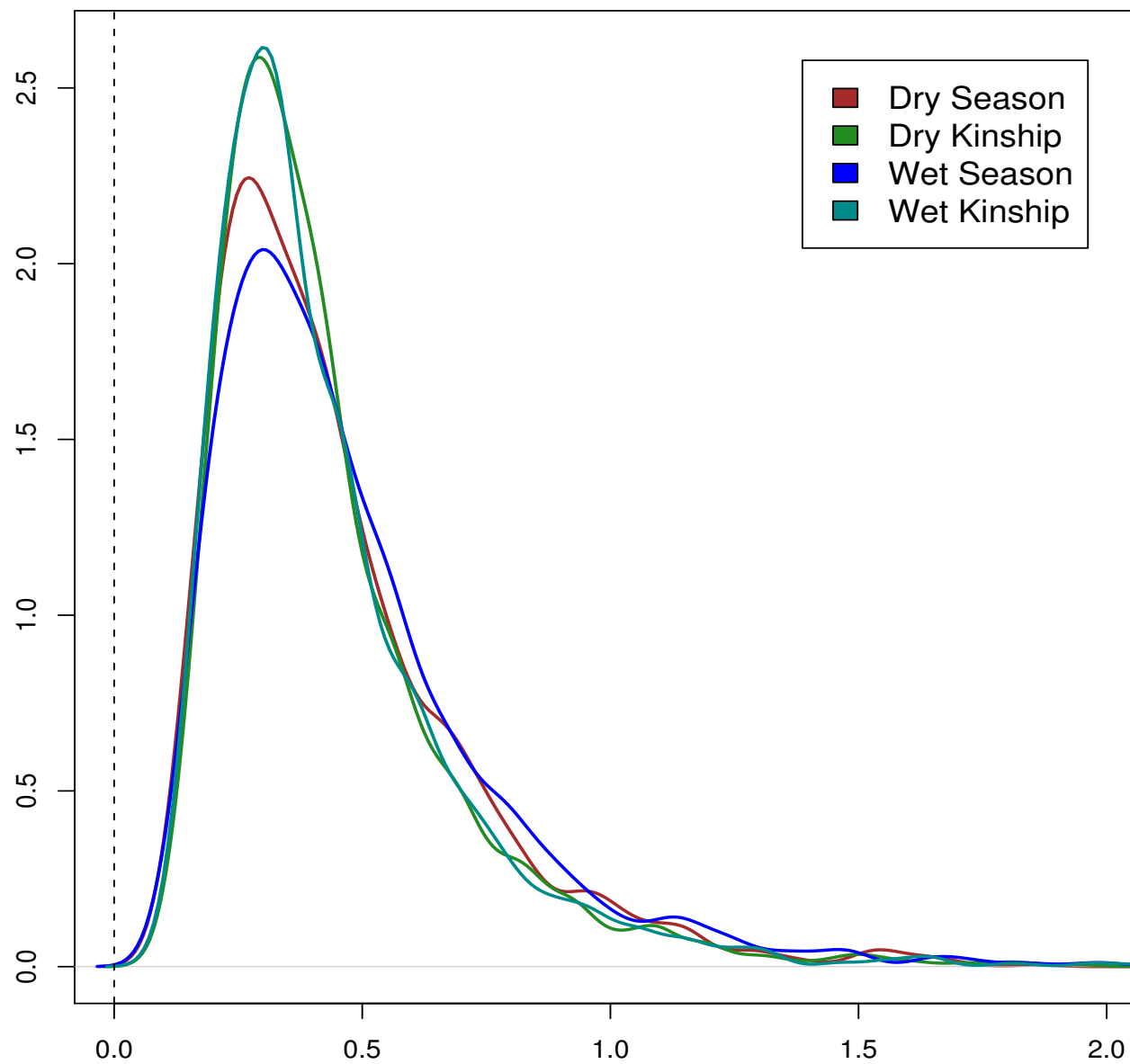
$$= \beta_0 + \beta_k k_{ij} + \epsilon_{ij}$$

$$\epsilon_{ij} = a_i + a_j + \gamma_{ij} + z'_i z_j$$

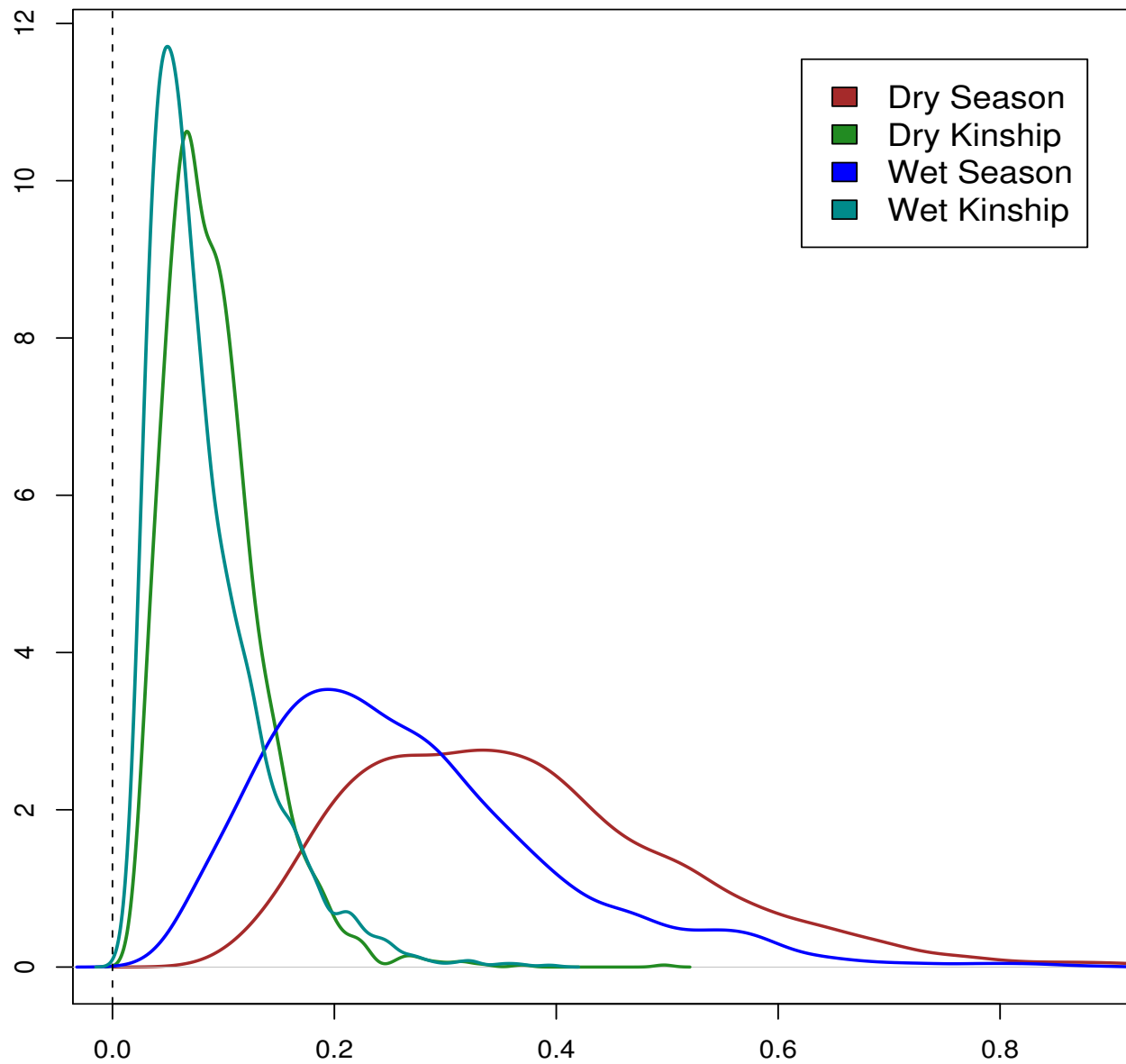
$$E(\epsilon_{ij}^2) = 2\sigma_a^2 + \sigma_\gamma^2 + \sigma_{z_1}^4 + \sigma_{z_2}^4$$



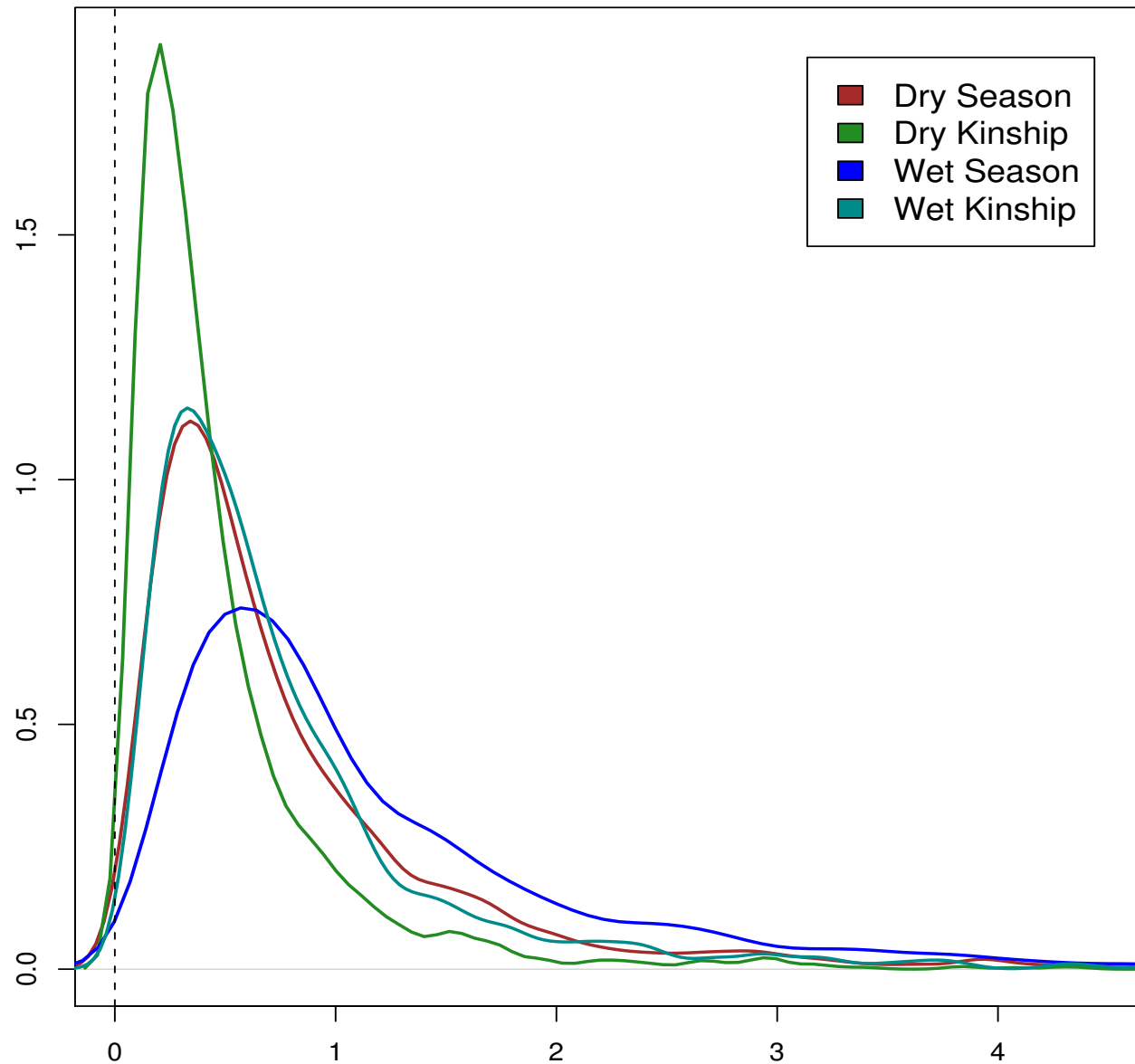
dukeblue **Posterior Sociability Variance  $2\sigma_a^2$**



dukeblue Posterior Normal Error  $\sigma_\gamma^2$

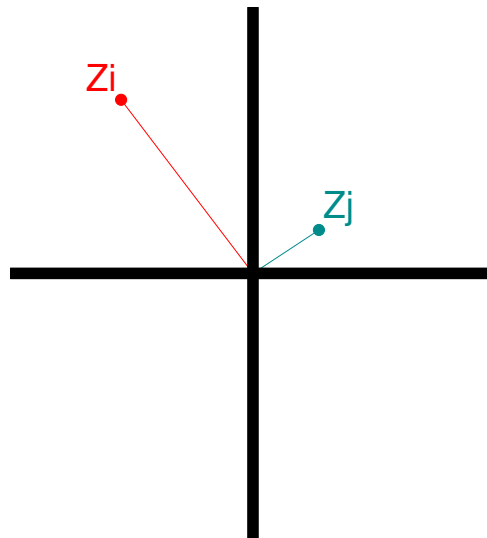


dukeblue **Posterior Social Space Variance** ( $\sigma_{z_1}^4 + \sigma_{z_2}^4$ )



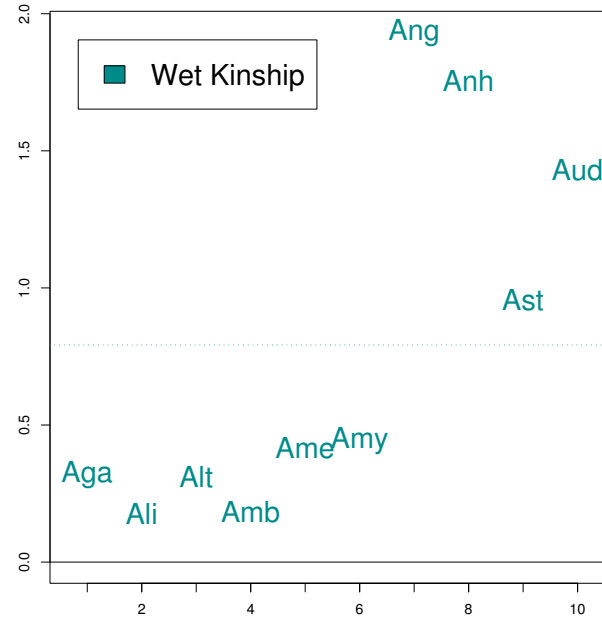
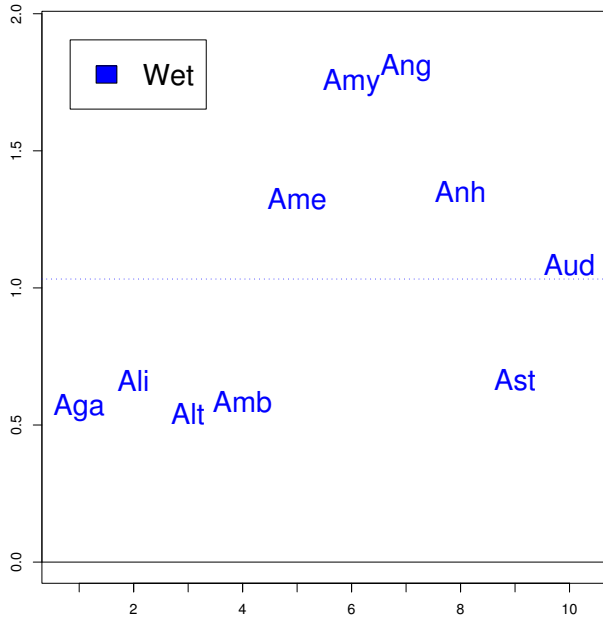
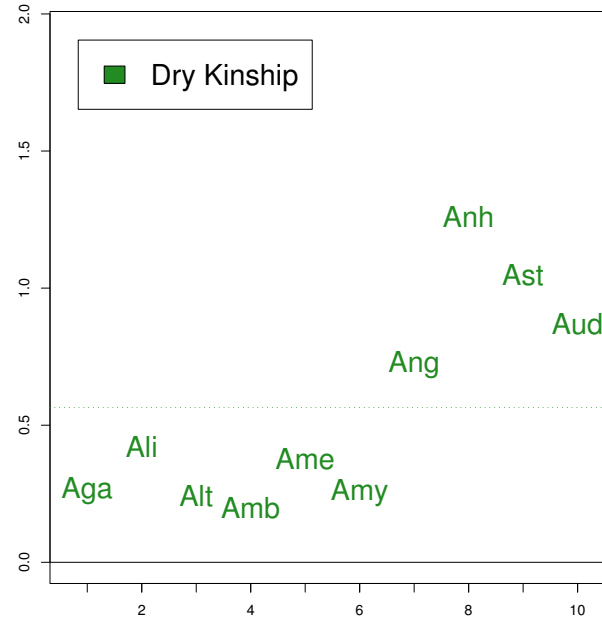
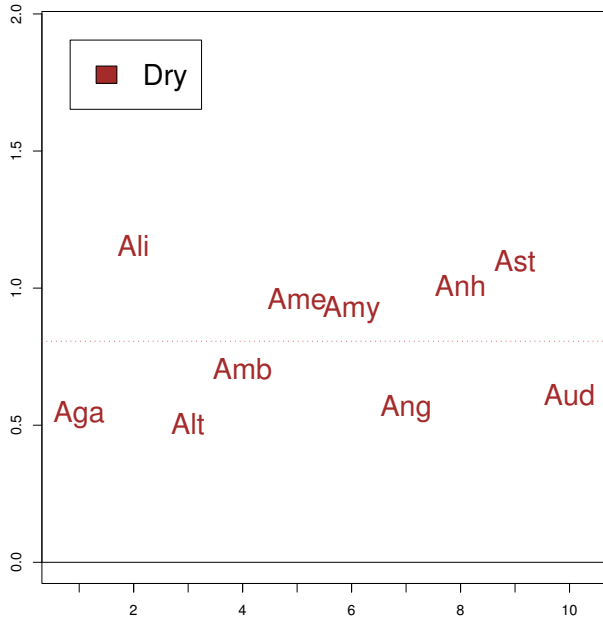
## dukeblue Pickiness in Social Space

- An elephant far away from the origin in social space tends to have highly positive or highly negative inner product pairwise terms in the model.
- Elephants close to the origin have small pairwise interaction effects.
- Pickiness is defined as the length of the vector in social space  $|\mathbf{z}_i|$ .



- Elephant  $i$  is “pickier” than elephant  $j$ .

# dukeblue Posterior Pickiness $|z_i|$



## dukeblue **Conclusions**

- The intercept  $\beta_0$  is greater in the Wet seasons than in the Dry seasons, indicating that the elephants are more gregarious during the Wet season.
- The role of kinships  $\beta_k$  is similar in the Wet and Dry season.
- Mother/Daughter relationships are significant.
- Sister relationships are also very important.
- After accounting for Mother/Daughter and Sister relationships, the DNA relatedness between elephants doesn't really matter.

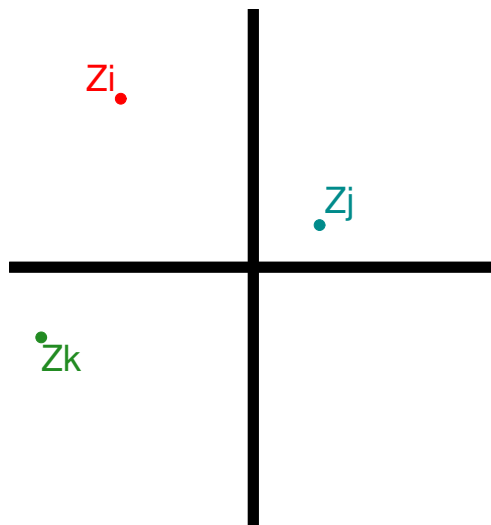
dukeblue Amy, Matriarch of Family AA



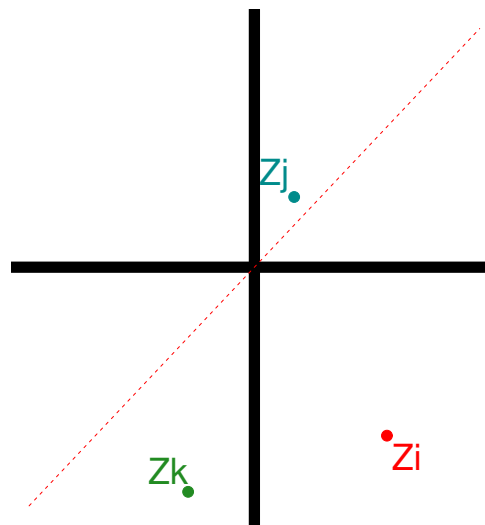
## dukeblue Pairwise Effects, Cont.

- Only the inner products  $z_i'z_j$ ,  $z_i'z_k$ ,  $z_j'z_k$  of the vectors  $z_i$ ,  $z_j$ ,  $z_k$  matter.

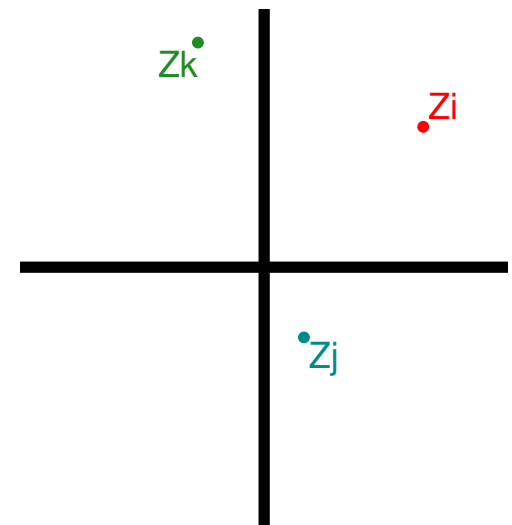
Reflections or rotations of social space do not change inner products.



Social Space



Reflection

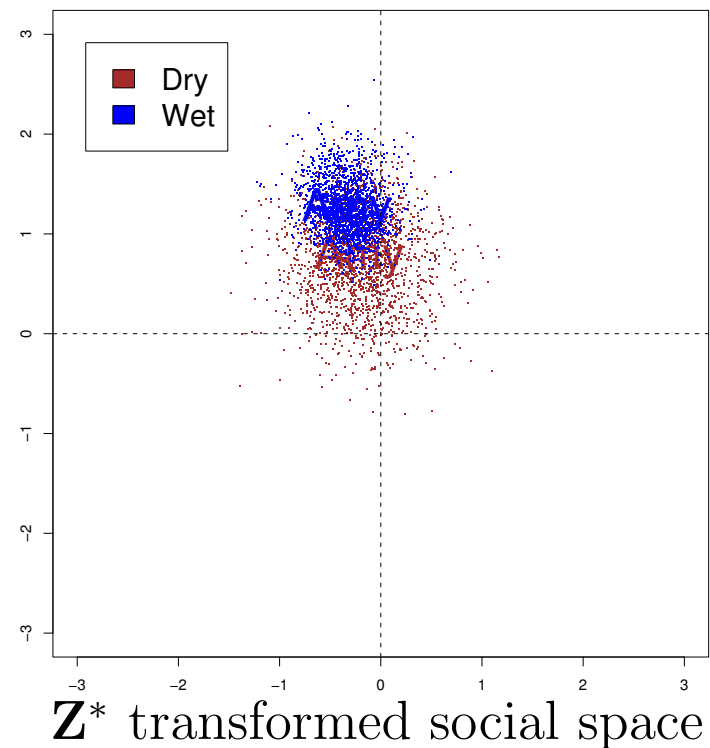
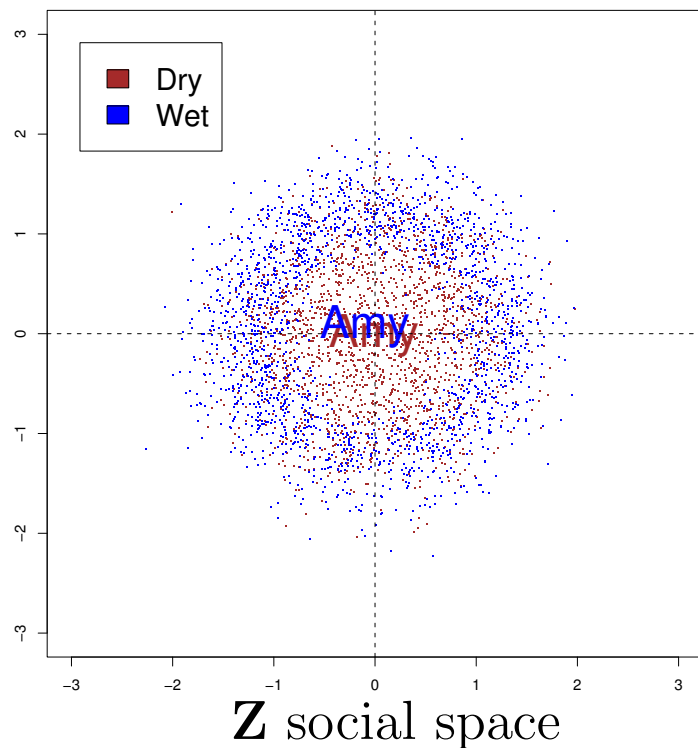


Rotation

- All 3 social spaces are equivalent.

## dukeblue Procrustean Transformation

- The posterior draws of the social space vectors must be reflected or rotated to give a coherent picture of the posterior distribution.



- Fix an arbitrary matrix  $Z_0$  of positions in social space, then apply the Procrustean transformation:

$$Z^* = Z_0 Z' (Z Z_0' Z_0 Z')^{-\frac{1}{2}} Z$$

## dukeblue Gibbs Sampling

$$\text{red}\theta_{ij} = \underbrace{\left(\frac{1}{2}\beta_0 + a_i\right)}_{\text{dukeblues}} + \underbrace{\left(\frac{1}{2}\beta_0 + a_j\right)}_{\text{dukeblues}} + \beta_k \mathfrak{g}_{ij} + \gamma_{ij} + z'_i z_j$$

**dukeblues** **dukebluer**

1. Sample linear effects:

- a) Sample  $\beta_d, s, r \mid \beta_0, \sigma_a^2, \sigma_\gamma^2, \theta, Z$ .
- b) Sample  $\beta_0 \mid s, r, \sigma_a^2$ .
- c) Sample  $\sigma_a^2, \sigma_\gamma^2 \mid \beta_0, s, r$ .

2. Sample bilinear effects:

- a) Sample  $z_i \mid Z_{-i}, \theta, \beta, s, r, \sigma_z^2, \sigma_\gamma^2$ .
- b) Sample  $\sigma_z^2 \mid Z \sim \text{IG}\left(\frac{1}{2} + \frac{nk}{2}, \frac{1}{2} + \frac{\text{tr}(Z'Z)}{2}\right)$ .

3. Update  $\theta_{ij}$  with a Metropolis step:

- a) Propose  $\theta_{ij}^* \sim \text{N}\left(\left(\frac{1}{2}\beta_0 + a_i\right) + \left(\frac{1}{2}\beta_0 + a_j\right) + \beta_k \mathfrak{k}_{ij} + z'_i z_j, \sigma_\gamma^2\right)$ .
- b) Accept  $\theta_{ij}^*$  with probability  $\left(\frac{p(y_{ij} \mid \theta_{ij}^*)}{p(y_{ij} \mid \theta_{ij})} \wedge 1\right)$ .

## dukeblue Selecting Dimension $k$ of Social Space

The method of choosing the dimension  $k$  of social space depends on the goal of the analysis.

### 1. Descriptive:

- Choose  $k = 2$  to give easily interpretable results.

### 2. Assessing model fit: how well does the model explain the data?

- Various model selection techniques

- My choice would be to use stochastic search variable selection with point-mass 0 mixture priors on  $\sigma_{z_1}^2, \sigma_{z_2}^2, \sigma_{z_3}^2, \dots$
- Requires proper priors.

### 3. Make predictions of unobserved data.

- Cross-validation techniques.

- Select  $k$  based on the predictive performance  $= \sum_{l=1}^4 \sum_{\{i,j\} \in A_l} \log p(y_{ij} | \hat{\theta}_{ij})$ , where  $\hat{\theta}_{ij}$  is the posterior mean excluding pairs in  $A_l$ .



dukeblue **Family AA Dry Observations**

DRY	Amy	Amy	Ang	Ang	Amb	Amb	Aud	Aud	Ali	Ali	Ast	Ast	Aga	Aga	Alt	Alt	Ame	Ame
	P	A	P	A	P	A	P	A	P	A	P	A	P	A	P	A	P	A
Amy P	272	0	237	7	245	2	245	8	154	64	147	61	215	54	205	54	182	68
Amy A	0	159	35	152	27	157	27	151	118	95	125	98	57	105	67	105	90	91
Ang P	237	7	244	0	224	23	217	36	154	64	142	66	189	80	179	80	162	88
Ang A	35	152	0	187	20	164	27	151	90	123	102	121	55	107	65	107	82	99
Amb P	245	2	224	23	247	0	220	33	153	65	141	67	201	68	194	65	174	76
Amb A	27	157	20	164	0	184	27	151	94	119	106	117	46	116	53	119	73	108
Aud P	245	8	217	36	220	33	253	0	149	69	145	63	205	64	197	62	170	80
Aud A	27	151	27	151	27	151	0	178	104	109	108	115	48	114	56	116	83	98
Ali P	154	64	154	64	153	65	149	69	218	0	191	17	147	122	144	115	115	135
Ali A	118	95	90	123	94	119	104	109	0	213	27	196	71	91	74	98	103	78
Ast P	147	61	142	66	141	67	145	63	191	17	208	0	148	121	145	114	121	129
Ast A	125	98	102	121	106	117	108	115	27	196	0	223	60	102	63	109	87	94
Aga P	215	54	189	80	201	68	205	64	147	122	148	121	269	0	257	2	193	57
Aga A	57	105	55	107	46	116	48	114	71	91	60	102	0	162	12	160	76	105
Alt P	205	54	179	80	194	65	197	62	144	115	145	114	257	2	259	0	185	65
Alt A	67	105	65	107	53	119	56	116	74	98	63	109	12	160	0	172	74	107
Ame P	182	68	162	88	174	76	170	80	115	135	121	129	193	57	185	65	250	0
Ame A	90	91	82	99	73	108	83	98	103	78	87	94	76	105	74	107	0	181

dukeblue **Kinship Relatedness**

AA	Amy	Angelina	Amber	Audrey	Alison	Astrid	Agatha	Althea	Amelia
Amy									
Angelina	0.39								
Amber	0.46	0.26							
Audrey	0.31	0.22	0.11						
Alison	0.3	0.02	0.08	0.02					
Astrid	0.25	0.18	0.02	0.14	0.53				
Agatha	0.27	0.08	0.25	-0.15	0.38	0.34			
Althea	0.05	0.05	0.12	-0.01	0.15	0.21	0.46		
Amelia	0.35	0.06	-0.09	0.01	0.09	0.2	0.1	0.02	
Anghared	0.25	0.34	0.02	0.2	-0.06	0.1	0.06	-0.04	0.52