Sta 205 : Home Work #10

Due : April 09, 2008

I. Convergence Of Series, Strong Law

(A) Let $S_n := \sum_{i=1}^n 1/i$. Explain carefully why S_n does or does not converge to a finite limit. Now let $S_n := \sum_{i=1}^n (-1^i)/i$. Does the sequence of real numbers S_n converge in this case to a finite limit? Why or why not? Finally let $\{X_n\}$ be a sequence of independent binary random variables with $\mathbb{P}(X_n = \pm 1) = \frac{1}{2}$. Does $S_n := \sum_{i=1}^n X_i/i$ converge to a real-valued random variable? In what sense? Why?

(B) The SLLN states that if $\{X_n, n \geq 1\}$ are iid with $\mathbb{E}|X_1| < \infty$, then

$$S_n/n \to \mathbb{E}(X_1) \qquad \text{a.s.}$$

Show that also

$$S_n/n \to \mathbb{E}(X_1) \qquad \text{in } L_1$$

(C) Define a sequence $\{X_n\}$ of random variables iteratively as follows. Let X_1 have a uniform distribution on $[0, 1]$, and for $n \geq 1$, let X_{n+1} have a uniform distribution on $[0, X_n]$. Show that

$$\frac{1}{n} \log X_n$$

converges a.s. and find the almost sure limit.

II. Two Statistical Concepts

(A) Let f_0 and f_1 be probability mass functions (pmfs) on the set $S := \{1, 2, ..., 100\}$, i.e., real valued functions satisfying, for $\theta \in \{0, 1\}$,

$$(\forall y \in S)\ f_\theta(y) \geq 0 \qquad \sum_{y \in S} f_\theta(y) = 1.$$

Let $\{X_n\}$ be iid random variables with pmf f_0, taking values in the set S, so $\Pr[X_n = y] = f_0(y)$ for $y \in S$. Set

$$Z_n := \prod_{i=1}^n \frac{f_1(X_i)}{f_0(X_i)}$$

Prove that $Z_n \to 0$ almost surely if $f_0(y) \neq f_1(y)$ for at least one $y \in S$. Be careful about any points where $f_0(y) = 0$ or $f_1(y) = 0$. Z_n is the *likelihood ratio* statistic, the basis for both Bayesian and classical tests of the "hypothesis" $H_0 :\ \theta = 0$ with alternative $H_1 :\ \theta = 1$.

(B) Suppose $g : (0,1] \mapsto \mathbb{R}$ is measurable and Lebesgue integrable. Let $\{U_n, n \geq 1\}$ be iid variables distributed uniformly on $(0,1]$ and define $X_i := g(U_i)$. In what sense does $I_n := \sum_{i=1}^n X_i/n$ approximate $I := \int_0^1 g(x)dx$? (This offers a way to approximate the integral by Monte Carlo methods). How large would n have to be to ensure that $\mathsf{P}[|I_n - I| > 0.10] < 0.01$? (You may have to make some additional assumptions, and the answer may depend on g— say what assumptions you made).