

Sta 205 : Homework #5

Due : February 20, 2008

1. Independence.

- (a) Let $\{B_i\}$, be independent events. For $N \in \mathbb{N}$ show that

$$\mathbb{P}\left(\bigcup_{i=1}^N B_i\right) = 1 - \prod_{i=1}^N [1 - \mathbb{P}(B_i)]$$

- (b) If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that

$$\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m) \quad \forall n, m \in \mathbb{N}, n \neq m,$$

does it follow that the events $\{A_n\}$ are independent? Give a proof or counter-example.

- (c) Let X be a random variable. Show that X is independent of itself if and only if there is some constant $c \in \mathbb{R}$ for which $\mathbb{P}[X = c] = 1$. Let f be a Borel measurable function, and X a random variable whose distribution is *not* concentrated at a single point. Can $f(X)$ and X be independent? Explain your answer.
- (d) Show that if the event A is independent of the π -system \mathcal{P} and $A \in \sigma(\mathcal{P})$, then $\mathbb{P}(A)$ is either 0 or 1.
- (e) Give a simple example to show that two random variables on the same space (Ω, \mathcal{F}) may be independent according to one probability measure \mathbb{P}_1 but dependent with respect to another \mathbb{P}_2 .

2. Borel Cantelli.

- (a) Fix any number $p > 0$ and let $\{X_n\}$ be a sequence of Bernoulli random variables with

$$\mathbb{P}(X_n = 1) = n^{-p} \quad \mathbb{P}(X_n = 0) = 1 - n^{-p}.$$

For $p = 2$ show that the partial sum

$$S_n := \sum_{k=1}^n X_k$$

converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, with $p = 1$, does S_n converge? Why or why not?

- (b) Dane draws independent random variables X_n from the uniform distribution on the unit interval $(0, 1]$. Each time he observes a “new record,” a random variable X_n larger than any of the previous observations $\{X_k : k < n\}$, he shouts “Whoo hoo!” Show that the probability of a Whoo Hoo shout on the n^{th} draw is $1/n$. Prove that Dane will say Whoo Hoo infinitely-many times. Do you need to assume that the events $A_n := \{ \text{Whoo Hoo on } n^{\text{th}} \text{ draw} \}$ are independent?

- (c) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Also show that

$$\mathbb{P}[X_n \text{ converges}] = 0$$

if the independent random variables are identically distributed with a non-trivial (i.e., not almost-surely constant) distribution.

- (d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$, there exists constants $c_n \rightarrow \infty$ such that

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Find the numbers c_n explicitly in terms of the CDF functions

$$F_n(x) = \mathbb{P}[X_n \leq x], \quad x \in \mathbb{R}.$$

Find a suitable sequence $\{c_n\}$ explicitly for an i.i.d. sequence $\{X_n\} \stackrel{\text{iid}}{\sim} \text{No}(0,1)$ of standard Gaussian random variables to ensure that $X_n/c_n \rightarrow 0$ almost surely.

3. Mixed Bag.

- (a) Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathbb{P}(A_n) < 1, \forall n \in \mathbb{N}$. Show that $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathbb{P}(A_n \text{ i.o.}) = 1$. Give an example to show that the condition $\mathbb{P}(A_n) < 1$ cannot be dropped.
- (b) Suppose $\{A_n\}$ is a sequence of events. If $\mathbb{P}(A_n) \rightarrow 1$ as $n \rightarrow \infty$, prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathbb{P}(\bigcap_k A_{n_k}) > 0$.
- (c) Let A_n be a sequence of events. If there exists $\epsilon > 0$ such that $\mathbb{P}(A_n) \geq \epsilon$ for all $n \in \mathbb{N}$, does it follow that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathbb{P}(\bigcap_k A_{n_k}) > 0$? Why or why not?
- (d) Let $\{X_n\}$ be i.i.d. random variables, with tail σ -field

$$\mathcal{T} \equiv \bigcap_n \mathcal{F}_n, \quad \mathcal{F}_n \equiv \sigma\{X_m : m \geq n\}$$

Is the event

$$\begin{aligned} E &= \{\text{There exists a number } B < \infty \text{ such that } |X_n| \leq B \text{ for infinitely-many } n\} \\ &= \bigcup_{B=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega)| \leq B\} \end{aligned}$$

in \mathcal{T} ? Prove or disprove it. Find the probability $\mathbb{P}[E]$ in terms of the random variables' common CDF, $F(x) \equiv \mathbb{P}[X_n \leq x]$.