# Sta 205 : Homework 2

**Due : January 30, 2008**

I. **$\sigma$ - algebras and Probability Assignments.**

(A) Let $\{A, B, C\} \subset \mathcal{F}$ be three events in a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Recall that a *partition* is a finite or countable collection of *disjoint* events $\Lambda_j \in \mathcal{F}$ with $\cup \Lambda_j = \Omega$. Enumerate all the elements of the partition $\mathcal{P} = \mathcal{P}(A, B, C)$ generated by these events (*i.e.*, $\mathcal{P}$ is the smallest partition for which $\{A, B, C\} \subset \sigma(\mathcal{P})$). How many (nonempty) elements does $\mathcal{P}$ have, at most? How many, at minimum?

(B) How many elements does the $\sigma$-algebra $\sigma(\mathcal{P})$ contain? Describe them in words (don't list them)

(C) Let's further assume that the above mentioned events $A, B, C$ are disjoint with probabilities $\mathsf{P}(A) = 0.6$, $\mathsf{P}(B) = 0.3$, $\mathsf{P}(C) = 0.1$. Calculate the probability of every event in $\sigma(A, B, C)$.

II. **Fun with null sets.**

(A) Let $\{A_n, n \in \mathbb{N}\}$ be events such that $\mathsf{P}(A_n) = 0,\ \forall n$. Show that $\mathsf{P}(\cup_{n=1}^{\infty} A_n) = 0$.

(B) Let $\{B_n, n \in \mathbb{N}\}$ be events such that $\mathsf{P}(B_n) = 1,\ \forall n \in \mathbb{N}$. What is $\mathsf{P}(\cap_{n=1}^{\infty} B_n)$ ?

(C) Now consider the set of events, $\{E_\alpha, \alpha \in \mathbb{R}\}$, such that $\mathsf{P}(E_\alpha) = 0, \forall \alpha \in \mathbb{R}$. Does it necessarily follow that $\mathsf{P}(\cup_{\alpha \in \mathbb{R}} E_\alpha) = 0$ ? If yes, give a proof, otherwise give a counter example.

(D) Finally, let $\{B_k\}$ be a collection of events such that, $\sum_{k=1}^{n} \mathsf{P}(B_k) > n - 1$. Show that $\mathsf{P}(\cap_{k=1}^{n} B_k) > 0$ for every $n \in \mathbb{N}$.

III. **Distribution functions and continuity.**

(A) Give an example of a function which in continuous on $\mathbb{R}$, but **not** uniformly continuous.

(B) Let $G$ be a continuous distribution function on $\mathbb{R}$. Show that $G$ is in fact uniformly continuous. Hint: Consider the points $\{x_i\}$ for which $G(x_i) = i/n$ for $0 < i < n$.

(C) Now let $F$ be any distribution function on $\mathbb{R}$. Show that $F$ can have **at most countably** many discontinuities. Hint: Consider the open intervals $(F(x-), F(x))$ for discontinuity points $x$.

IV. $\pi$ & $\lambda$ **- systems.**

(A) Let $\Omega = (0,1] \times (0,1]$, and consider the following collections of subsets of $\Omega$:

$$\mathcal{A} = \{(0,a] \times (0,b] : \ 0 < a, b \leq 1\}$$

i. Is $\mathcal{A}$ a $\pi$ - system? Why or why not?

ii. Is $\mathcal{A}$ a $\lambda$ - system? Why or why not?

(B) Consider the following collecton of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b], \ b \in \mathbb{R}\}$$

i. Show that $\mathcal{B}$ is a $\pi$ - system, but not a $\lambda$ system.

ii. What is the $\lambda$ - system generated by $\mathcal{B}$?

V. $\pi$ **- systems and fields.**

(A) Let $\mathcal{C}$ be a non empty collection of subsets of $\Omega$, and let $\mathcal{A}(\mathcal{C})$ be the minimal field over $\mathcal{C}$. Show that $\mathcal{A}(\mathcal{C})$ consists of sets of the form

$$\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{ij},$$

where for each pair $(i,j)$ either $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and where the $m$ sets $\{B_i := \cap_{j=1}^{n_i} A_{ij}, \ 1 \leq i \leq m\}$, are disjoint. Thus, we can represent explicitly the sets in $\mathcal{A}(\mathcal{C})$, however it turns out that, we cannot do the same for the $\sigma$-field over $\mathcal{C}$.

(B) Now let's further assume that $\mathcal{C}$ is a $\pi$ system. Show that if $\mathsf{P}_1$, $\mathsf{P}_2$ are two probability measures which agree on $\mathcal{C}$, then $\mathsf{P}_1$, $\mathsf{P}_2$ must also agree on $\mathcal{A}(\mathcal{C})$. Hint: Use part(A) and the inclusion-exclusion principle.

(C) Find two probability measures $\mathsf{P}_1$, $\mathsf{P}_2$ on some set $\Omega$ that agree on a collection of subsets $\mathcal{C}$, but *not* on $\mathcal{A}(\mathcal{C})$. Obviously (from the previous part) $\mathcal{C}$ cannot be a $\pi$-system. Hint: It's enough to have $\mathcal{C} = \{A, B\}$ with just two elements, on an outcome space $\Omega$ with just three points.