Midterm Examination

STA 215: Statistical Inference

Wednesday, 2007 Mar 7, 1:15-2:30 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
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5.	/20
Total:	/100

Problem 1. For some real number $\theta \in \Theta = (0, \infty)$ let i.i.d. random variables X_j all have C.D.F.

$$F_\theta(x) := \mathsf{P}_\theta[X_j \le x] = \left(\frac{x}{x+1}\right)^\theta, \quad 0 < x < \infty$$

a. (4) Find the likelihood function for a vector of n observations, $\mathbf{x}_n = \{X_j : 1 \le j \le n\}$. Simplify!

$f(\mathbf{x}_n \mid \theta) =$ ______________________________

b. (4) Find the maximum likelihood estimator $\hat{\theta}_n$ for these observations:

$\hat{\theta}(\mathbf{x}_n) =$ ______________________________

Problem 1 (cont).

c. (4) Find the Fisher information $I(\theta)$ for one observation:

$I(\theta) =$ ______________________

d. (4) Is this an exponential family? ◯ Yes ◯ No
If so, specify η, T, and B for the standard form

$$f(x \mid \theta) = e^{\eta(\theta)\cdot T(x) - B(\theta)}\, h(x)$$

with the smallest suitable $q \in \mathbb{N}$, $\eta(\theta) \in \mathbb{R}^q$, $T(x) \in \mathbb{R}^q$, $B(\theta) \in \mathbb{R}$, and $h(x) \geq 0$; if not, explain why.

e. (4) For which $p \in \mathbb{R}$ is $\mathsf{E}_\theta[X^p] < \infty$? Why? You need not evaluate the expectations.

Problem 2. The random variables $\{Y_i\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, \theta)$ are independent, all from the uniform distribution on the interval $[0, \theta]$ for some $\theta \in \Theta \subset \mathbb{R}_+$. Unfortunately we observe only n truncated versions,

$$X_i = \min(Y_i, 1),\ 1 \le i \le n.$$

Let $\mathbf{x}_n = \{X_1, \ldots, X_n\}$ be the observation vector.

a. (4) For $\Theta = (1, \infty)$, find the likelihood function:

$f(\mathbf{x}_n \mid \theta) =$ ______________________

b. (4) Find a sufficient statistic $S(\mathbf{x}_n)$ and the Maximum Likelihood Estimator $\hat{\theta}(\mathbf{x}_n)$, still with $\Theta = (1, \infty)$:

$S(\mathbf{x}_n) =$ ______________________

$\hat{\theta}(\mathbf{x}_n) =$ ______________________

Problem 2 (cont).

c. (4) Still with $\Theta = (1, \infty)$, is this an exponential family? ◯ Yes ◯ No
If so, specify η, T, and B for the standard form

$$f(x \mid \theta) = e^{\eta(\theta)\cdot T(x) - B(\theta)}\, h(x)$$

with the smallest suitable $q \in \mathbb{N}$, $\eta(\theta) \in \mathbb{R}^q$, $T(x) \in \mathbb{R}^q$, $B(\theta) \in \mathbb{R}$, and $h(x) \geq 0$; if not, explain why.

d. (4) Now find the M.L.E. for $\Theta = (0, \infty)$, rather than $(1, \infty)$ as above:

$$\hat{\theta}(\mathbf{x}_n) = \begin{cases} & \\ & \\ & \end{cases}$$

e. (4) Is your statistic $S(\mathbf{x}_n)$ (from b.) still sufficient with $\Theta = (0, \infty)$?
◯ Yes ◯ No Explain.

Problem 3. For some fixed $\alpha > 0$ let $X \sim \mathsf{Pa}(\alpha, 1)$ be a Pareto random variable, satisfying

$$\mathsf{P}[X > x] = \begin{cases} x^{-\alpha} & x > 1 \\ 1 & x \le 1. \end{cases}$$

a. (4) For each $p \in \mathbb{R} = (-\infty, \infty)$, evaluate $\mathsf{E}[X^p]$. For which powers p is $\mathsf{E}[X^p] < \infty$?

$\mathsf{E}[X^p] =$

b. (4) Give the Method of Moments estimator for α, based on a random sample $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha, 1)$ of size n.

Problem 3 (cont).

c. (4) What is the natural sufficient statistic $T(x)$ for this natural exponential family?

$T(x) =$

d. (4) Find the Jeffreys prior density function for α:

$\pi_J(\alpha) =$ ______________________

e. (4) Find the probability distribution of $T(X)$. For which powers $p \in \mathbb{R}$ (positive *or* negative) is $\mathsf{E}[|T(X)|^p] < \infty$?

Problem 4. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ge}(p)$ have the geometric distribution with

$$\mathsf{P}[X_i = x] = p\, q^x, \quad x \in \{0, 1, ...\}$$

where $p \in (0, 1)$ and $q := (1-p)$.

a. (5) Find the Fisher Information for a single observation:

$I(p) =$ ____

b. (5) Find the Maximum Likelihood Estimator for a sample of size $n \in \mathbb{N}$, $\mathbf{x}_n = \{X_1, ..., X_n\}$:

$\hat{p}(\mathbf{x}_n) =$ ____

Problem 4 (cont).

c. (5) Write $\hat{p}(\mathbf{x}_n)$ as a function $g(\bar{\mathbf{x}}_n)$ of $\bar{\mathbf{x}}_n = \frac{1}{n}\Sigma X_j$ and use the "delta method" to find the approximate mean and variance of $\hat{p}(\mathbf{x}_n)$. Is $\hat{p}(\mathbf{x}_n)$ unbiased? ◯ Yes ◯ No Asymptotically unbiased? ◯ Yes ◯ No

$\mathsf{E}[\hat{p}(\mathbf{x}_n)] \approx$ ______________ $\mathsf{V}[\hat{p}(\mathbf{x}_n)] \approx$ ______________

d. (5) Verify that $\hat{p}(\mathbf{x}_n)$ is asymptotically efficient, *i.e.*, has asymptotic squared-error risk $\lim_{n\to\infty} n\mathsf{V}[\hat{p}(\mathbf{x}_n)]$ equal to the Information Inequality lower bound.

Problem 5. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Po}(\lambda)$ have the Poisson distribution with

$$\mathsf{P}[X_i = x] = \lambda^x \, e^{-\lambda}/x!, \quad x \in \{0, 1, ...\}$$

where $\lambda \in \mathbb{R}_+$.

a. (5) Find the (single-observation) Fisher Information for the natural parameter η of this exponential family:

$I(\eta) =$ ____

b. (5) Find the Maximum Likelihood Estimator for a sample of size $n \in \mathbb{N}$, $\mathbf{x}_n = \{X_1, ..., X_n\}$:

$\hat{\eta}(\mathbf{x}_n) =$ ____

Problem 5 (cont).

c. (5) Give an approximate 90% confidence interval for η on the basis of a sample of size $n = 100$ with $S_n := \sum_{j=1}^{n} X_j = 900$, based on the asymptotic normality of the natural sufficient statistic. Recall that $\Phi(1.282) \approx 0.90$, $\Phi(1.645) \approx 0.95$.

d. (5) Find the (exact) Bayesian posterior mean for λ (not η), using the Jeffreys prior π_J:

$$\mathsf{E}_J[\lambda \mid S_{100} = 900] = \underline{\hspace{6cm}}.$$

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedekor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_1}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		