

Sta 205 : Homework #10

Due : April 11, 2007

I. Convergence In Distribution

- (A) For events $\{A_n\}$ and A in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define Bernoulli random variables by $X_n \equiv 1_{A_n}$ and $X \equiv 1_A$. As $n \rightarrow \infty$,
- Under what conditions on $\{A_n\}$ and A will $X_n \Rightarrow X$?
 - Under what conditions on $\{A_n\}$ and A will $X_n \rightarrow X$ in L_1 ?
 - Under what conditions on $\{A_n\}$ and A will $X_n \rightarrow X$ in L_∞ ?
- (B) Let $\{X_n\}$ be a sequence of RV's with distributions given by

$$\mathbb{P} \left[X_n = 1 - \frac{1}{n} \right] = \mathbb{P} \left[X_n = 1 + \frac{1}{n} \right] = \frac{1}{2}.$$

Show that X_n converges in distribution, and find the limiting distribution.

- (C) Define probability density functions by

$$f_n(x) = \begin{cases} 1 - \cos(2n\pi x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and let F_n be the corresponding distribution functions. Show that F_n converges weakly and find the weak limit. Also show that the density functions f_n do *not* converge pointwise.

- (D) Let $Y_n \sim \text{No}(\mu_n, \sigma_n^2)$ and $Y \sim \text{No}(\mu, \sigma^2)$ be normally-distributed random variables. Show that $Y_n \Rightarrow Y$ if and only if $\mu_n \rightarrow \mu$ and $\sigma_n^2 \rightarrow \sigma^2$.

II. Central Limit Theorem (CLT)

- (A) Fix $a > 1$ and let X_n be an i.i.d. sequence with density function

$$f(x) = a|x|^{-1-2a}, \quad |x| \geq 1; \quad f(x) = 0, \quad |x| < 1.$$

Compute $\mathbb{E}[X_1]$ and $\mathbb{E}[X_1^2]$. Set $S_n \equiv \sum_{i=1}^n X_i$. Find the limiting distributions of S_n/n and of S_n/\sqrt{n} as $n \rightarrow \infty$. Extra credit: What happens for $a < 1$? For $a = 1$?

- (B) **Delta method.** Let $\{X_j\} \stackrel{\text{iid}}{\sim} \text{Bi}(1, \theta)$ be independent Bernoulli random variables with partial sum $S_n \equiv \sum_{j \leq n} X_j \sim \text{Bi}(n, \theta)$ and sample mean $\bar{X}_n \equiv S_n/n$, for some $\theta \in (0, 1)$, and let $\phi \in \mathcal{C}^2(0, 1)$ be a twice-differentiable real-valued function on the unit interval. For large n find the approximate mean and variance of $\phi(\bar{X}_n)$, correct to order $1/n$. Show your work; keep track of the error terms!