

Sta 205 : Home Work #8

Due : March 28, 2007

I. Convergence Concepts: a.s. and i.p.

(A) Let $\{X_n\}$ be a monotonically increasing sequence of RVs such that $X_n \rightarrow X$ in probability (i.p). Show that $X_n \rightarrow X$ almost surely (a.s).

(B) Let $\{X_n\}$ be any sequence of RVs. Show that $X_n \rightarrow X$ a.s. if and only if

$$\sup_{k \geq n} |X_k - X| \rightarrow 0 \quad \text{i.p.}$$

(C) Let $\{X_n\}$ be an arbitrary sequence of RVs and set $S_n = \sum_{i=1}^n X_i$. Show that $X_n \rightarrow 0$ a.s. implies that $S_n/n \rightarrow 0$ a.s.

(D) Let $\{X_n\}, \{Y_n\}$ be two sequences of RVs such that $0 \leq X_n \leq Y_n$ and $Y_n \rightarrow 0$ i.p. Show that $X_n \rightarrow 0$ i.p.

(E) Suppose $\{X_n\}$ are identically distributed with finite variance. Fix $\epsilon > 0$. Show that $n\mathbf{P}\left[|X_1| \geq \epsilon\sqrt{n}\right] \rightarrow 0$. Also show that $\frac{\sum_{i=1}^n |X_i|}{\sqrt{n}} \rightarrow 0$ i.p.

(F) For random variables X, Y define

$$\rho(X, Y) \equiv \inf\{\delta > 0 : \mathbf{P}[|X - Y| \geq \delta] \leq \delta\}.$$

The function ρ is a metric (you do not have to prove that), i.e., it's non-negative, symmetric, satisfies the triangle inequality, and vanishes if and only if $X = Y$. Show that ρ "metrizes" convergence in probability: i.e, $X_n \rightarrow X$ i.p., if and only if $\rho(X_n, X) \rightarrow 0$.

II. L_p Convergence

(A) Let $\{X_n\}$ be a sequence of positive RVs such that $X_n \rightarrow X$ i.p. and $\mathbf{E}(X_n) \rightarrow \mathbf{E}(X)$. Show that $X_n \rightarrow X$ in L_1 .

(B) For any two events A and B , define the distance $d(A, B)$ as

$$d(A, B) \equiv \mathbf{P}(A \Delta B)$$

where (as usual) $A \Delta B \equiv (A \cap B^c) \cup (A^c \cap B)$ denotes the symmetric difference. Prove that for any sequence of events $\{A_n\}$, $d(A_n, A) \rightarrow 0$ if and only if $\mathbf{1}_{A_n} \rightarrow \mathbf{1}_A$ in L_2 .

- (C) Give an example of a sequence of RVs which converge in L_1 but do not converge in L_2 . Let $\{X_n\}$ be a sequence of RVs such that $X_n \rightarrow X$ in L_1 , $X \in L_2$. Give a simple condition on $\{X_n\}$ (which can be easily verified) such that $X_n \rightarrow X$ in L_2 . Justify your answer.
- (D) Let $(\Omega, \mathcal{F}, \mathbf{P}) = ([0, 1], \mathcal{B}[0, 1], \lambda)$ be the unit interval with Lebesgue measure, and define $X_n(\omega) \equiv \omega^n$, $\omega \in [0, 1]$. For what $p \in [1, \infty]$, does the sequence $\{X_n\}$ converge in L_p ? If it does converge for some $p \in [1, \infty]$, find the limiting random variable (it might depend on p). Explain your answer.

III. Two Miscellaneous (But Interesting) Concepts

(A) Egoroff:

- i. Suppose $\{X_n\}, X$ are real valued RVs defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that $X_n(\omega) \rightarrow X(\omega)$ for all ω in some event $\Lambda \in \mathcal{F}$. Show that for every $\epsilon > 0$, there is an event Λ_ϵ with $\mathbf{P}(\Lambda_\epsilon) < \epsilon$ and

$$\sup_{\Lambda \setminus \Lambda_\epsilon} |X(\omega) - X_n(\omega)| \rightarrow 0 \quad (n \rightarrow \infty)$$

Thus the convergence is uniform except on a small set. (For more on this result, called Egoroff's Theorem, see page 90 of the text.)

- ii. Now use Egoroff's theorem to show the following: Suppose $\{X_n\}$ be RVs such that $\sup_n |X_n| \leq K < \infty$ a.s. and $X_n \rightarrow X$ a.s. Show that $\mathbf{E}(X_n) \rightarrow \mathbf{E}(X)$. Notice that the last statement is an easy consequence of Dominated convergence theorem, but the point here is to use Egoroff!

(B) L_p Spaces :

- i. For a random variable X , $1 < p < q < \infty$, show that

$$0 \leq \|X\|_1 \leq \|X\|_p \leq \|X\|_q \leq \|X\|_\infty$$

- ii. For $1 < p < q < \infty$, show that

$$L_\infty \subset L_q \subset L_p \subset L_1$$

where L_p denotes the space of all RVs X with $\|X\|_q < \infty$. Hint: Jensen's inequality might be needed here.

- iii. Show the following form of Hölder's inequality : For RVs X, Y

$$\mathbf{E}(|XY|) \leq \|X\|_1 \|Y\|_\infty$$

- iv. Show the following form of Minkowski's inequality: For RVs X, Y

$$\|X + Y\|_\infty \leq \|X\|_\infty + \|Y\|_\infty$$