

# Sta 205 : Home Work #7

Due : March 07, 2007

I. **True or False?** Answer whether each of the following statements is true or false. If your answer is true, answer why it is true. If it is false, show why— perhaps by giving a simple counter example.

- (A) If  $\{X_n, n \in \mathbb{N}\}$  is a uniformly integrable (U.I.) collection of random variables, then  $X_n \in L_1$  for each  $n$ .
- (B) Define a sequence  $\{X_n\}$  of random variables on the unit interval with Lebesgue measure,  $(\Omega, \mathcal{F}, P)$  with  $\Omega = (0, 1]$ ,  $\mathcal{F} = \mathcal{B}$ , and  $P = \lambda$ , by  $X_n \equiv \sqrt{n} \mathbb{1}_{(0, \frac{1}{n}]}$ . Then  $\{X_n\}$  is UI.
- (C) Let  $\{X_n\}$  be a sequence of random variables for which  $e^{|X_n|}$  is uniformly bounded in  $L_1$ , i.e., satisfies  $\mathbb{E}e^{|X_n|} \leq B$  for some  $B < \infty$  and all  $n$ . Then  $\{X_n\}$  is UI.
- (D) Let  $\{X_n\}$  be a sequence of random variables that is uniformly bounded in  $L_1$ , i.e., satisfies  $\mathbb{E}|X_n| \leq B$  for some  $B < \infty$  and all  $n$ . Then  $\{X_n\}$  is UI.

## II. Characteristic Functions.

(A) Let  $X$  be a random variable, and define

$$\phi_X(\omega) \equiv \mathbb{E}(e^{i\omega X}), \quad \omega \in \mathbb{R}$$

Show that  $\phi_X(\omega)$  is uniformly in continuous in  $\mathbb{R}$ .

(B) Find the characteristic functions of the following random variables :

- i.  $X \sim \text{Ge}(p)$ <sup>1</sup>
- ii.  $Y \sim \text{Ex}(\lambda)$ <sup>2</sup>
- iii.  $Z = X/n, \quad X \sim \text{Ge}(\lambda/n)$

Find the limit of  $\phi_Z(\omega)$  from part (iii) above as  $n \rightarrow \infty$ . Recognize it?

## III. Infinite Divisibility.

The distribution of a random variable  $X$  is called *infinitely divisible* if, for every  $n \in \mathbb{N}$ , there exist  $n$  i.i.d random variables  $\{Y_i\}$  such that  $X$  has the same distribution as  $\sum_{i=1}^n Y_i$ . Use characteristic functions to show that if  $X \sim \text{Po}(\lambda)$ , then  $X$  is infinite divisible. (Hint: Recall that if random variables  $\{Y_i\}$  are independent then  $\phi_{\sum Y_i}(\omega) = \prod \phi_{Y_i}(\omega)$  for all  $\omega \in \mathbb{R}$ )

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<sup>1</sup>Starting at  $x = 0$ — with p.m.f.  $f(x | p) = pq^x, x = 0, 1, 2, \dots$

<sup>2</sup>Rate parametrization— with p.d.f.  $f(y) = \lambda e^{-\lambda y}, y > 0$ .