

Sta 205 : Home Work #6

Due : 2007 Feb 28

1. Fubini and Tonelli.

- (a) Let X be a positive random variable (i.e, $X \geq 0$ a.s). Show that

$$\mathbf{E}(X) = \int_0^\infty \mathbf{P}(X > t) dt$$

(note X need not have an absolutely-continuous distribution). Also verify that for any $\alpha > 0$

$$\mathbf{E}(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} \mathbf{P}(X > t) dt$$

- (b) Define probability spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, for $i = 1, 2$ as follows. Let each $\Omega_i := (0, 1)$, the unit interval, with σ -algebras

$$\mathcal{B}_1 = \text{Borel sets of } (0,1) \quad \mathcal{B}_2 = \text{All subsets of } (0,1),$$

and let μ_1 be Lebesgue measure and μ_2 counting measure— so that $\mu_1(A)$ is the length of any set $A \in \mathcal{B}_1$ and $\mu_2(A)$ is the number of points in $A \in \mathcal{B}_2$. Define

$$f(x, y) = \mathbf{1}_{x=y}(x, y)$$

Set

$$I_1 = \int_{\Omega_1} \left[\int_{\Omega_2} f(x, y) \mu_2(dy) \right] \mu_1(dx) \quad I_2 = \int_{\Omega_2} \left[\int_{\Omega_1} f(x, y) \mu_1(dx) \right] \mu_2(dy)$$

Compute I_1 and I_2 . Is $I_1 = I_2$? Are the measures μ_1 and μ_2 σ -finite? Why doesn't Fubini's theorem hold here?

- (c) This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose F and G are two distribution functions with no common points of discontinuity in an interval $(a, b]$. Show that

$$\int_{(a,b]} G(x) F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)$$

Show that the above formula need not hold true if F and G have common discontinuities.

2. Uniform Integrability (UI).

- (a) Let $\{X_n\}$ be a sequence of iid, L_1 random variables. Set $S_n \equiv \sum_{i=1}^n X_i$. Show that the sequence of random variables $\{Y_n\}$ defined by $Y_n \equiv S_n/n$ is UI.
- (b) Let $X_n \sim \text{No}(0, \sigma_n^2)$. Find a simple (easily verifiable) condition on $\{\sigma_n^2\}$ such that $\{X_n\}$ is UI.
- (c) If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.
- (d) Suppose $\{X_n, n \geq 1\}$ is an **arbitrary** sequence of non-negative random variables, and set $M_n \equiv \bigvee_{i=1}^n X_i$. If $\{X_n\}$ is UI, show that $\mathbf{E}(M_n)/n \rightarrow 0$.
- (e) Let $\phi(x)$ be a function which grows faster than x at infinity, i.e, $\phi(x)/x \rightarrow \infty$ as $x \rightarrow \infty$. Let \mathcal{C} be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$\mathbf{E}(\phi(|Z|)) \leq B.$$

Show that \mathcal{C} is UI.

3. Convergence Theorems Revisited.

- (a) Let X be a non-negative real valued random variable. Show that:
 - i. $\lim_{n \rightarrow \infty} n\mathbf{E}(\frac{1}{X}\mathbf{1}_{[X>n]}) = 0$.
 - ii. $\lim_{n \rightarrow \infty} n^{-1}\mathbf{E}(\frac{1}{X}\mathbf{1}_{[X>n^{-1}]}) = 0$.
- (b) Suppose $\{p_k, k \geq 0\}$ is a probability mass function on $\{0, 1, \dots\}$ and define the generating function

$$P(z) = \sum_{k=0}^{\infty} p_k z^k \quad 0 \leq z \leq 1$$

Prove using Dominated Convergence theorem that

$$\frac{d}{dz}P(z) = \sum_{k=1}^{\infty} p_k k z^{k-1} \quad 0 \leq z \leq 1.$$

Note you may wish to consider the cases $z < 1$ and $z = 1$ separately. What is $P'(1)$? $P'(0)$? Can you express the variance of X , if it exists, in terms of $P(z)$ and its derivatives?