# Sta 205 : Homework #5

**Due : February 21, 2007**

1. **Indepedence**.

   (a) Let $\{B_i\}$, be independent events. For $N \in \mathbb{N}$ show that

   $$\mathsf{P}\left(\bigcup_{i=1}^{N} B_i\right) = 1 - \prod_{i=1}^{N}[1 - \mathsf{P}(B_i)]$$

   (b) If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that

   $$\mathsf{P}(A_n \cap A_m) = \mathsf{P}(A_n)\mathsf{P}(A_m) \; \forall n, m \in \mathbb{N}, \; n \neq m,$$

   does it follow that the events $\{A_n\}$ are independent? Give a proof or counter-example.

   (c) Let $X$ be a random variable. Show that $X$ is independent of itself if and only if there is some constant $c \in \mathbb{R}$ for which $\mathsf{P}[X = c] = 1$. Let $f$ be a Borel measurable function, and $X$ a (not necessarily constant) random variable. Can $f(X)$ and $X$ be independent? Explain your answer.

   (d) Show that if the event $A$ is independent of the $\pi$-system $\mathcal{P}$ and $A \in \sigma(\mathcal{P})$, then $\mathsf{P}(A)$ is either 0 or 1.

   (e) Give a simple example to show that two random variables on the same space $(\Omega, \mathcal{F})$ may be independent according to one probability measure $\mathsf{P}_1$ but dependent with respect to another $\mathsf{P}_2$.

2. **Practice with Borel Cantelli**.

   (a) Let $\{X_n\}$ be a sequence of Bernoulli random variables with

   $$\mathsf{P}(X_n = 1) = n^{-p} \qquad \mathsf{P}(X_n = 0) = 1 - n^{-p}$$

   for some $p > 0$. For $p = 2$ show that the partial sum

   $$S_n := \sum_{k=1}^{n} X_k$$

   converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, with $p = 1$, does $S_n$ converge? Why or why not?

(b) Dane tosses a heavily biased coin repeatedly, with independent outcomes. He is convinced that if he chooses the probability of heads $p$ to be small enough (say, $p \approx 10^{-6}$), then only finitely-many heads will ever appear. Is Dane right? Justify your answer.

(c) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Let $\{X_n\}$ be a sequence of i.i.d. non-trivial (*i.e.*, not almost-surely constant) random variables, then show that

$$\mathsf{P}[X_n \text{ converges}] = 0$$

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$ , there exists constants $c_n \to \infty$ such that

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Give a careful description of how you choose $c_n$. Find a suitable sequence $\{c_n\}$ explicitly for an i.i.d. sequence $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ of standard Gaussian random variables to ensure that $X_n/c_n \to 0$ almost surely.

3. **Mixed Bag**.

(a) Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathsf{P}(A_n) < 1$, $\forall n \in \mathbb{N}$. Show that $\mathsf{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathsf{P}(A_n \text{ i.o.}) = 1$. Give an example to show that the condition $\mathsf{P}(A_n) < 1$ cannot be dropped.

(b) Suppose $\{A_n\}$ is a sequence of events. If $\mathsf{P}(A_n) \to 1$ as $n \to \infty$, prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$.

(c) Let $A_n$ be a sequence of events. If there exists $\epsilon > 0$ such that $\mathsf{P}(A_n) \geq \epsilon$ for all $n \in \mathbb{N}$, does it follow that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$? Why or why not?

(d) Let $\{X_n\}$ be i.i.d. random variables, with tail $\sigma$-field

$$\mathcal{T} \equiv \bigcap_n \mathcal{F}_n, \qquad \mathcal{F}_n \equiv \sigma\{X_m : m \geq n\}$$

Is the event

$$\begin{aligned} E &= \{\text{There exists a number } B < \infty \text{ such that } |X_n| \leq B \text{ for infinitely-many } n\} \\ &= \cup_{B=1}^{\infty} \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{\omega : \ |X_n(\omega)| \leq B\} \end{aligned}$$

in $\mathcal{T}$? Prove or disprove it. Find the probability $\mathsf{P}[E]$ in terms of the random variables' common CDF, $F(x) \equiv \mathsf{P}[X_n \leq x]$.