Sta 205 : Home Work #4

Due : February 14, 2007

1. **Expectation**.

   (a) Consider the triangle with vertices $(-1,0),(1,0),(0,1)$ and suppose (X_1, X_2) is a random vector uniformly distributed with in this triangle. Compute $\mathsf{E}(X_1 + X_2)$.

   (b) Let $((0,1], \mathcal{B}((0,1]), \lambda)$ be a probability space (λ denotes Lebesgue measure). Let X be a random variable defined on the probability space described above, with $X(\omega) = 1$, if $\omega \in \mathbb{Q}$ and 0 otherwise. What is $\mathsf{E}(X)$? Prove it.

   (c) Suppose $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$. Show that

   $$\int_{|X|>n} X d\mathsf{P} \to 0$$

   as n tends to ∞.

   (d) Let $\{A_n\}$ denote a sequence of events such that $\mathsf{P}(A_n) \to 0$ and let $X \in L_1$. Show that

   $$\int_{A_n} X d\mathsf{P} \to 0$$

   (e) Let $X \in L_1$, and let A be an event. Show that

   $$\int_A |X| d\mathsf{P} = 0 \text{ iff } \mathsf{P}(A \cap [|X| > 0]) = 0$$

   (f) Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $A_n \in \mathcal{F},\ n \in \mathbb{N}$. Define a distance measure $d : \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}_+$ by $d(A_n, A_m) \equiv \mathsf{P}(A_n \Delta A_m)$. Show that, if $\{A_n\} \subset \mathcal{F},\ A \in \mathcal{F}$ satisfy $d(A_n, A) \to 0$, then

   $$\int_{A_n} X dp \to \int_A X dp$$

   for every $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$. Note: Here "$\Delta$" denotes the symmetric set difference, $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$.

2. **Convergence Theorems**.

   (a) Let $X \geq 0$ be a non-negative random variable and define a sequence of real numbers by:

   $$S_n \equiv \sum_{k=0}^{\infty} \frac{k}{2^n} \mathsf{P}\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) \qquad n \in \mathbb{N}.$$

   What is $\lim_{n\to\infty} S_n$? Justify your answer.

   (b) Define a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathsf{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$ by

   $$X_n \equiv \frac{n}{\log n} \mathbf{1}_{(0,\frac{1}{n})} \qquad n \in \mathbb{N}.$$

   Show that $X_n \to 0$ almost surely, and $\mathsf{E}(X_n) \to 0$. Also show that the dominated convergence theorem does not apply to this example. Why?

   (c) Suppose $\{Y_n\}$ be a sequence of random variables such that

   $$\mathsf{P}(Y_n = \pm n^3) = \frac{1}{2n^2}, \qquad \mathsf{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

   Show using the Borel-Cantelli lemma that $Y_n \to 0$ almost surely. Compute $\lim_{n\to\infty} \mathsf{E}(Y_n)$. It is 0? Is the Lebesgue Dominated convergence theorem applicable? Why or why not?

   (d) Let $\{X_n\}, X$ be random variables, and $0 \leq X_n \to X$. If $\sup_n \mathsf{E}(X_n) \leq K < \infty$, then show that $\mathsf{E}(X) \leq K$ and $X \in L_1$.

3. **Potpourri**.

   (a) Let $\{X_n\}$ be a sequence of Bernoulli random variables with

   $$\mathsf{P}(X_n = 1) = p_n = 1 - \mathsf{P}(X_n = 0)$$

   Show that $\sum_{n=1}^{\infty} p_n < \infty$ implies $\sum_{n=1}^{\infty} \mathsf{E}(X_n) < \infty$ and hence conclude that $X_n \to 0$ almost surely.

   (b) Let $\{X_n\}$ be a sequence of random variables. Show that

   $$\mathsf{E}\left(\sup_{1 \leq n \leq \infty} |X_n|\right) < \infty$$

   if and only if there exists a random variable $0 \leq Y \in L_1$ such that

   $$\mathsf{P}(|X_n| \leq Y) = 1, \qquad \forall n \geq 1$$