Sta 205 : Homework #3

Due : February 07, 2007

I. **Random variables**.

(A) Let $(\Omega, \mathcal{B}, \mathsf{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$ for Lebesgue measure λ. Define:

$$\begin{aligned} X_1(\omega) &\equiv 0, \ \forall \omega \in \Omega \\ X_2(\omega) &\equiv \mathbf{1}_{\{1/2\}}(\omega) \\ X_3(\omega) &\equiv \mathbf{1}_{\mathbb{Q}}(\omega) \end{aligned}$$

where $\mathbb{Q}$ is set of rational numbers in $(0,1]$. Let $f(k) \equiv \mathsf{P}[X_1 = X_2 = X_3 = k]$, $k \in \mathbb{R}$. Plot the function $f(\cdot)$. What are the σ - algebra's generated by X_1, X_2 and X_3?

(B) Toss two independent fair coins and define a random variable ζ as follows: If both the coins show Heads, $\zeta = 1$; if both the coins show Tails $\zeta = 2$; and if the two coins differ, $\zeta = 0$. Suggest a suitable probability space $(\Omega, \mathcal{F}, \mathsf{P})$ for this problem— specify a sample space Ω, write ζ explicitly as a function on Ω, describe the σ-algebra $\mathcal{F}$ generated by ζ, and give the probability assignment on $\mathcal{F}$ for fair, independent coins. Is $\mathcal{F}$ identical to the power set $\mathcal{P} := \{A : \ A \subset \Omega\}$ of Ω? Is ζ measurable with respect to $\mathcal{P}$?

II. **More on Random variables**.

(A) Let X be a random variable with CDF $F(x) := \mathsf{P}(X \le x)$. Set $Y \equiv F(X)$. If X has a continuous distribution, show that Y is a random variable and that Y has a uniform distribution on $[0,1]$.

(B) If X is a real valued random variable (so $\mathsf{P}[|X| < \infty] = 1$), then show that for any $\epsilon > 0$, there exists a *bounded* random variable Y such that

$$\mathsf{P}(X \ne Y) < \epsilon$$

(A random variable Y is bounded if it satisfies $|Y(\omega)| \le K_\epsilon$ for some fixed number $K_\epsilon < \infty$ that does not depend on ω.)

III. **Measurable functions**.

(A) Let $\Omega = \mathbb{R}$, and set $\mathcal{S} = \{\Omega, \emptyset, (-\infty, 0], (0, \infty)\}$. Show that $\mathcal{S}$ is a σ- algebra. What functions f defined on Ω are $\mathcal{S}$ measurable ?

(B) If X is a random variable, then show that so is $|X|$. Show by an example that the converse need not be true.

(C) Let $\Omega = \mathbb{R}$. Let $\mathcal{S}_0 \equiv \{\emptyset, \Omega\}$ be the so-called trivial σ-algebra. Consider the function $X(\omega) = \omega^2 : \Omega \to \mathbb{R}$. Is the function X $\mathcal{S}_0 \backslash \mathcal{B}$ measurable? Justify your answer. Find the σ-algebra $\mathcal{F}_X := \sigma(X)$ generated by X. Is the set $(-\infty, 0]$ in $\mathcal{F}_X$? How about $[-4, 4]$?

(D) Let $\{X_n, n \geq 0\}$ be real-valued random variables on the probability space $(\Omega, \mathcal{B}, \mathsf{P})$ that satisfy

$$\lim_{n\to\infty} X_n(\omega) = +\infty$$

for every $\omega \in \Omega$, and let $B < \infty$ be a real number. Consider the integer-valued quantities

$$\tau(\omega) \equiv \inf\{n \geq 0 : X_n(\omega) \geq B\}$$

Prove that τ is a random variable.

Extra credit: Prove that X_τ is also a random variable.

IV. **Practice with limsup and liminf**.

(A) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on Ω and set $Y \equiv \limsup X_n$. For $\beta \in \mathbb{R}$, express the event $\{\omega \in \Omega : Y(\omega) \leq \beta\}$ in terms of unions, intersections, and complements of events of the form $\{\omega \in \Omega : X_n(\omega) \leq \beta\}$. Why is Y a random variable?

(B) Now set $Z \equiv \liminf X_n$. For $\alpha \in \mathbb{R}$, express the event $\{\omega \in \Omega : Z(\omega) \leq \alpha\}$ in terms of unions, and intersections, and complements of events of the form $\{\omega \in \Omega : X_n(\omega) \leq \alpha\}$. Why is Z a random variable ?