

# Sta 205 : Homework 2

Due : January 31, 2007

## I. $\sigma$ - algebra and probability assignments.

- (A) Let  $\{A, B, C\} \subset \mathcal{F}$  be three events in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Recall that a *partition* is a finite or countable collection of *disjoint* events  $\Lambda_j \in \mathcal{F}$  with  $\cup \Lambda_j = \Omega$ . Enumerate all the elements of the partition  $\mathcal{P} = \mathcal{P}(A, B, C)$  generated by these events (*i.e.*,  $\mathcal{P}$  is the smallest partition for which  $\{A, B, C\} \subset \sigma(\mathcal{P})$ ). How many (nonempty) elements does  $\mathcal{P}$  have, at most? How many, at minimum?
- (B) How many elements does the  $\sigma$ -algebra  $\sigma(\mathcal{P})$  contain? Describe them in words (don't list them)
- (C) Let's further assume that the above mentioned events  $A, B, C$  are disjoint with probabilities  $\mathbb{P}(A) = 0.6$ ,  $\mathbb{P}(B) = 0.3$ ,  $\mathbb{P}(C) = 0.1$ . Calculate the probability of every event in  $\sigma(A, B, C)$ .

## II. Practice with null sets.

- (A) Let  $\{A_n, n \in \mathbb{N}\}$  be events such that  $\mathbb{P}(A_n) = 0$ ,  $\forall n$ . Show that  $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = 0$ .
- (B) Let  $\{B_n, n \in \mathbb{N}\}$  be events such that  $\mathbb{P}(B_n) = 1$ ,  $\forall n \in \mathbb{N}$ . What is  $\mathbb{P}(\cap_{n=1}^{\infty} B_n)$  ?
- (C) Now consider the set of events,  $\{E_\alpha, \alpha \in \mathbb{R}\}$ , such that  $\mathbb{P}(E_\alpha) = 0, \forall \alpha \in \mathbb{R}$ . Does it necessarily follow that  $\mathbb{P}(\cup_{\alpha \in \mathbb{R}} E_\alpha) = 0$  ? If yes, give a proof, otherwise give a counter example.
- (D) Finally, let  $\{B_k\}$  be a collection of events such that,  $\sum_{k=1}^n \mathbb{P}(B_k) > n - 1$ . Show that  $\mathbb{P}(\cap_{k=1}^n B_k) > 0$ .

## III. Distribution functions and continuity.

- (A) Give an example of a function which is continuous on  $\mathbb{R}$ , but **not** uniformly continuous.
- (B) Let  $G$  be a continuous distribution function on  $\mathbb{R}$ . Show that  $G$  is in fact uniformly continuous. Hint: Consider the points  $\{x_i\}$  for which  $G(x_i) = i/n$  for  $1 \leq i < n$ .
- (C) Now let  $F$  be any distribution function on  $\mathbb{R}$ . Show that  $F$  can have **at most countably** many discontinuities. Hint: Consider the open intervals  $(F(x-), F(x))$  for discontinuity points  $x$ .

#### IV. $\pi$ & $\lambda$ - systems.

(A) Let  $\Omega = (0, 1] \times (0, 1]$ , and consider the following collections of subsets of  $\Omega$ :

$$\mathcal{A} = \{(0, a] \times (0, b] : 0 < a, b \leq 1\}$$

- i. Is  $\mathcal{A}$  a  $\pi$  - system? Why or why not?
- ii. Is  $\mathcal{A}$  a  $\lambda$  - system? Why or why not?

(B) Consider the following collection of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b], b \in \mathbb{R}\}$$

- i. Show that  $\mathcal{B}$  is a  $\pi$  - system, but not a  $\lambda$  system.
- ii. What is the  $\lambda$  - system generated by  $\mathcal{B}$ ?

#### V. $\pi$ - systems and fields.

(A) Let  $\mathcal{C}$  be a non empty collection of subsets of  $\Omega$ , and let  $\mathcal{A}(\mathcal{C})$  be the minimal field over  $\mathcal{C}$ . Show that  $\mathcal{A}(\mathcal{C})$  consists of sets of the form

$$\cup_{i=1}^m \cap_{j=1}^{n_i} A_{ij},$$

where for each  $i, j$  either  $A_{ij} \in \mathcal{C}$  or  $A_{ij}^c \in \mathcal{C}$ , and where the  $m$  sets  $\cap_{j=1}^{n_i} A_{ij}$ ,  $1 \leq i \leq m$ , are disjoint. Thus, we can explicitly represent the sets in  $\mathcal{A}(\mathcal{C})$ , however it turns out that, we cannot do the same for the  $\sigma$ -field over  $\mathcal{C}$ .

- (B) Now let's further assume that  $\mathcal{C}$  is a  $\pi$  system. Show that if  $P_1, P_2$  are two probability measures which agree on  $\mathcal{C}$ , then  $P_1, P_2$  must also agree on  $\mathcal{A}(\mathcal{C})$ . Hint: Use part(A) and the inclusion-exclusion principle.
- (C) Find two probability measures  $P_1, P_2$  on some set  $\Omega$  that agree on a collection of subsets  $\mathcal{C}$ , but *not* on  $\mathcal{A}(\mathcal{C})$ . Obviously (from the previous part)  $\mathcal{C}$  cannot be a  $\pi$ -system. Hint: It's enough to have  $\mathcal{C} = \{A, B\}$  with just two elements, on an outcome space  $\Omega$  with just three points.