

Sta 205 : Homework 1

Due : January 24, 2007

I. Fields and σ - fields.

- (A) For a three-point outcome set $\Omega = \{a, b, c\}$ and $\mathcal{C} := \{\{a\}\}$, enumerate the class \aleph of all σ -fields \mathcal{F} on Ω that contain \mathcal{C} , *i.e.*, satisfy $\mathcal{C} \subset \mathcal{F}$. Also find $\sigma(\mathcal{C})$.
- (B) For each integer $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, set

$$A_n := \left\{ \frac{m}{n} : m \in \mathbb{N} \right\}.$$

Find $\limsup_{n \rightarrow \infty} A_n$ and $\liminf_{n \rightarrow \infty} A_n$.

- (C) Let f and $\{f_n\}$ be real-valued functions on any set Ω , and let ϵ_k be any decreasing sequence such that $\epsilon_k \searrow 0$ as $k \rightarrow \infty$. Show that:

$$\{\omega : f_n(\omega) \rightarrow f(\omega)\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega : |f_n(\omega) - f(\omega)| < \epsilon_k\}.$$

- (D) Find a set Ω and two fields \mathcal{F}_1 and \mathcal{F}_2 for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a field.
- (E) Suppose a collection $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of σ -fields satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\cup \mathcal{F}_j$ is a field? (the answer is “yes”— show why)
- (F) Under the same conditions, is $\cup \mathcal{F}_j$ a σ -field? (this one is “no”— find a counter-example. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\cup_n A_n \notin \mathcal{F}_j$ for every j , hence $\cup_n A_n \notin \cup_j \mathcal{F}_j$).
- (G) Let $\Omega := (0, 1]$ be the half-open unit interval, and let $n \in \mathbb{N}$ be a FIXED positive integer (say, three). Set

$$\mathcal{B}_n := \{(0, j/2^n], j \in \{0, 1, \dots, 2^n\}\},$$

the collection of half-open intervals from zero up to an integral multiple of 2^{-n} . Describe in words the σ -field

$$\mathcal{F}_n := \sigma(\mathcal{B}_n)$$

generated by \mathcal{B}_n . Tell how many elements \mathcal{B}_n has, and how many elements \mathcal{F}_n has.

II. Asymptotic Density.

For any subset $A \subset \mathbb{N}$ of positive integers write $\#(A)$ for its cardinality, *i.e.*, the number of elements it contains. The set A has “asymptotic density” $d(A)$ if the limit

$$d(A) := \lim_{n \rightarrow \infty} \frac{\#(A \cap \{1, 2, \dots, n\})}{n} \quad (1)$$

exists (*i.e.*, the lim-sup and lim-inf coincide). Denote by

$$\mathcal{A} := \{A \subset \mathbb{N} : d(A) \text{ exists} \}$$

the collection of all subsets of \mathbb{N} that have an asymptotic density.

- (A) Show carefully that the set of even numbers $E := 2\mathbb{N} = \{2, 4, \dots\}$ has asymptotic density $d(E) = 1/2$. By “carefully” I mean: compute the ratio in Equation (1) *exactly*, for every n (consider even and odd n separately), and compute the limit.
- (B) Find a set $B \subset \mathbb{N}$ that does *not* have an asymptotic density.
- (C) Find a set $A \in \mathcal{A}$ for which $A \cap E \notin \mathcal{A}$, and conclude that \mathcal{A} is not a field. Hint: the on-line lecture notes might help if you get stuck.