Final Examination

STA 205: Probability and Measure Theory

Due by Thursday, 2007 May 3, 7:00 pm

This is an open-book take-home examination. You may work on it during any consecutive 48-hour period you like; please record your starting and ending times on the lines below.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible.

This exam is due by 7pm Thursday, 2007 May 3. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name:	____________________
Issued:	4:00 , Apr 27 , 2007
Started:	: , , 2007
Finished:	: , , 2007
Due by:	7:00 , May 3 , 2007

1.	/20
2.	/20
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Total:	/120

Problem 1: Let $X_1, X_2, \ldots$ and Y be random variables that satisfy the condition

$$\mathsf{P}(|X_n| \le c) \;\ge\; \mathsf{P}(Y \le c) \tag{1}$$

for each $n \in \mathbb{N} = \{1, 2, \ldots\}$ and each $c > 0$.

a) Show that $\{X_n\}$ is uniformly integrable if $Y \in L_1$.
(Hint: Without loss of generality you may assume each $X_n \ge 0$ and $Y \ge 0$ (why?). Use the fact that the expectation of any positive random variable $X \ge 0$ is $\mathsf{E}X = \int_0^\infty \mathsf{P}[X > x]\,dx$.)

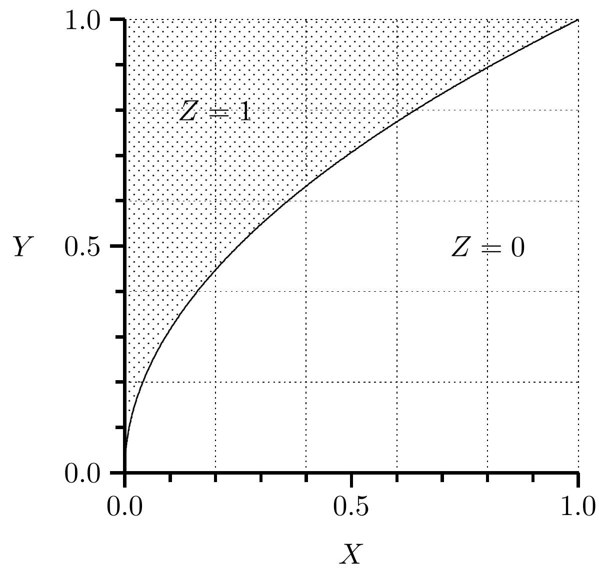
b) If a collection $\{X_n\}$ of random variables is *dominated* by Y, *i.e.*, if $|X_n| \le Y$ almost surely, does (1) follow? Give a proof or counter-example.

c) Does (1) imply that $|X_n| \le Y$ almost surely? Give a proof or counter-example.

Problem 2: On the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where P is Lebesgue measure on the Borel sets $\mathcal{F}$ in the unit square $\Omega := \{(x,y) : \ 0 \le x, y < 1\}$, define four random variables by:

$$W(x,y) := \sqrt{x} \qquad X(x,y) := x \qquad Y(x,y) := y \qquad Z(x,y) := \mathbf{1}_{\{x<y^2\}}$$

Here's a graphical representation:



a) Give the distribution (by name, including parameters) and expectation for each of these. For example, either "$X \sim \mathsf{Un}(0,1)$" or "$X \sim \mathsf{Be}(1,1)$" would be correct, with $\mathsf{E}[X] = 1/2$; what about W and Z?

$W \sim$ ______________ $\mathsf{E}[W] =$ ______________

$X \sim$ ______________ $\mathsf{E}[X] =$ ______________

$Y \sim$ ______________ $\mathsf{E}[Y] =$ ______________

$Z \sim$ ______________ $\mathsf{E}[Z] =$ ______________

Problem 2 (cont'd):

b) Give each of the conditional expectations requested:

$$\mathsf{E}[X \mid Y, Z] = \begin{cases} \\ \\ \end{cases} \qquad \mathsf{E}[X \mid Z] = \begin{cases} \\ \\ \end{cases}$$

$$\mathsf{E}[Z \mid X] = \qquad \mathsf{E}[Z \mid Y] =$$

c) Give each of the conditional probabilities, for $0 < t < 1$:

$$\mathsf{P}[X \le t \mid Z] = \begin{cases} \\ \\ \end{cases} \qquad \mathsf{P}[X \le t \mid Y, Z] = \begin{cases} \\ \\ \end{cases}$$

$$\mathsf{P}[X \le t \mid Y] = \begin{cases} \\ \\ \end{cases} \qquad \mathsf{P}[X \le Y \mid W] = \begin{cases} \\ \\ \end{cases}$$

Problem 3: Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ex}(\lambda)$ for some fixed $\lambda > 0$. Set $Z_0 = 1$ and, for $n \in \mathbb{N}$, $Z_n := \prod_{i=1}^n X_i$.

a) Show that for any $p > 0$ the random variables $Y_i := (X_i)^p$ are independent with the $Y_i \overset{\text{iid}}{\sim} \mathsf{We}(\alpha, \beta)$ Weibull distribution for some parameters $\alpha, \beta \in \mathbb{R}_+$. Find α, β, and $\mathsf{E}[Y_i]$ in terms of λ and p.

$\alpha =$ ____________ $\beta =$ ____________ $\mathsf{E}[Y_i] =$ ____________

b) For which $\lambda > 0$ and $p > 0$ is $\{M_n := (Z_n)^p\}$ a martingale?

c) For $\lambda = \sqrt[3]{6}$, find the best bound you can for

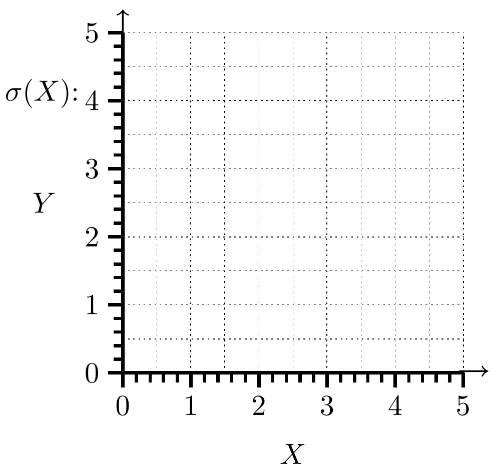
$$\mathsf{P}[\sup_{0 \le n < \infty} Z_n \ge c] \le \text{____________________}$$

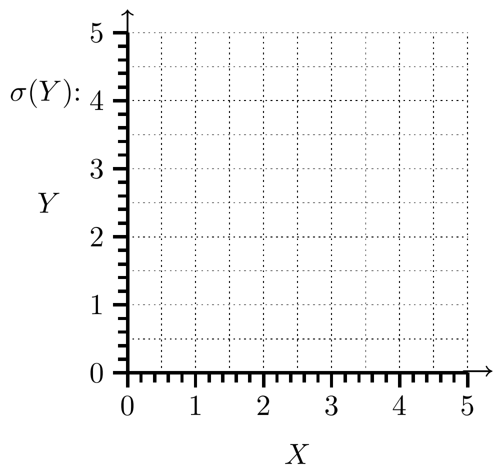
for all $c > 1$ (the bound will depend on c of course).

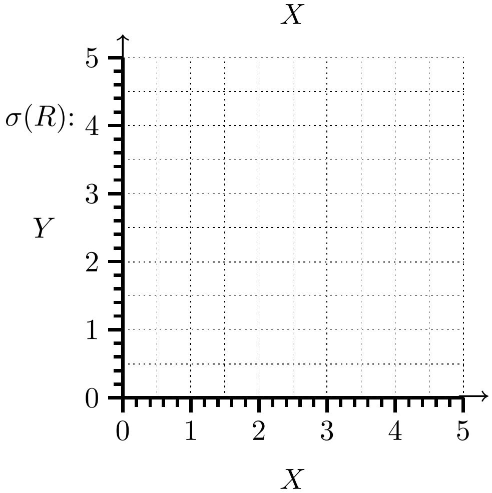
Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space with $\Omega = \mathbb{R}^2_+$, Borel sets $\mathcal{F}$, and $\mathsf{P}(dx, dy) := e^{-x-y}\, dx\, dy$ and construct four random variables

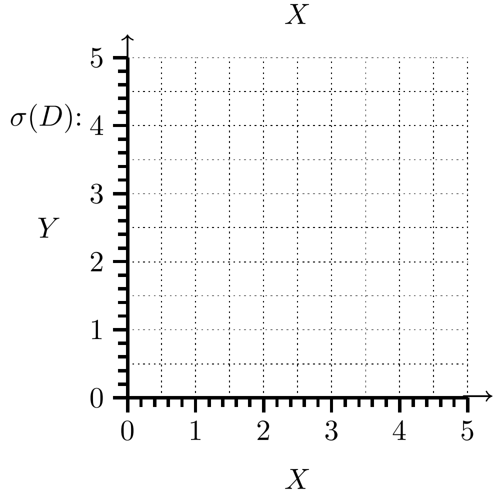
$$X(x,y) := x; \quad Y(x,y) := y; \quad R(x,y) := y/x; \quad D(x,y) := y - x.$$

a) Each of the squares below represents (the bottom left corner[1] of) Ω. Sketch an event in each of $\sigma(X)$, $\sigma(Y)$, $\sigma(R)$, and $\sigma(D)$ (respectively) that is *not* in any of the other three σ-algebras:









[1] Only the portion $[0,5)^2$ of the unbounded set $\Omega = \mathbb{R}^2_+$ would fit on the paper, but that's enough for this problem.

Problem 4 (cont'd): Still $R = Y/X$, $D = Y - X$ for $X, Y \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$:

b) Find the marginal density functions (correctly throughout $\mathbb{R}$):

$f_X(x) =$ ______________ $f_Y(y) =$ ______________

$f_R(r) =$ ______________ $f_D(d) =$ ______________

c) Find the joint density functions (correctly throughout $\mathbb{R}^2$):

$f_{XR}(x, r) =$ ______________ $f_{XD}(x, d) =$ ______________

Problem 4 (cont'd): Still $R = Y/X$, $D = Y - X$ for $X, Y \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$:

d) Find the conditional density functions (correctly throughout $\mathbb{R}^2$):

$f_{X|R}(x \mid r) =$ ______________ $f_{X|D}(x \mid d) =$ ______________

e) Identify the conditional distributions for X given from 4d) above by name and parameter values for the specific values of $r = 1$ and $d = 0$. Can you explain what these represent geometrically in Ω? Why do they differ?

Problem 5: For $n \in \mathbb{N}$ and $a \in \mathbb{R}$ define random variables

$$X_n(\omega) := 2^{an}\, \mathbf{1}_{\{\omega \le 2^{-n}\}}$$

on the unit interval $\Omega = (0, 1]$ with Lebesgue measure $\mathsf{P}(d\omega)$ on the Borel sets $\mathcal{F}$. Set

$$S_n := \sum_{j=1}^{n} X_j,$$

the sequence of partial sums.

a) For which (if any) numbers $a \in \mathbb{R}$ does S_n converge almost surely as $n \to \infty$? Why?

b) For which (if any) numbers $a \in \mathbb{R}$ does S_n converge in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ as $n \to \infty$? Why?

c) For which (if any) numbers $a \in \mathbb{R}$ is X_n a martingale? Why?

d) For which (if any) numbers $a \in \mathbb{R}$ is $\{X_n\}$ Uniformly Integrable? Why?

e) For which (if any) numbers $a \in \mathbb{R}$ is $\{X_n\}$ dominated by some $Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$? Why? Specify a suitable Y.

Problem 6: Skyler is trying to find $\mu := \mathsf{E}[X]$ for $X \sim \mathsf{Be}(4,1)$. Too lazy to compute the integral (or look on the attached pdf table, where he'd find that it is $\mu = 4/5$), he decides to estimate the integral by Monte Carlo importance sampling. He fixes some $\theta > 0$ (more about choosing θ below) and draws independent samples $x_i \sim \mathsf{Be}(\theta,1)$, with probability density functions $\Pi(x) = \theta x^{\theta-1}\mathbf{1}_{\{0<x<1\}}$, and evaluates the weighted average

$$I_n := \frac{1}{n}\sum_{j=1}^{n} g(x_j)\; w_j = \frac{1}{n}\sum_{j=1}^{n} x_j\; \frac{4\,(x_j)^{4-1}}{\theta\,(x_j)^{\theta-1}}$$

of the function $g(x) := x$ along the sequence $\{x_j\} \overset{\text{iid}}{\sim} \mathsf{Be}(\theta,1)$, with weights given by the pdf ratio $w_j := \frac{f(x_j|4,1)}{f(x_j|\theta,1)}$.

a) Show that $I_n \to \mu$ almost surely as $n \to \infty$, for any $\theta > 0$.

b) Compute the mean and variance of I_n, for every $\theta > 0$.

c) What value of θ will minimize $\mathsf{E}\big[|I_n - \mu|^2\big]$?

d) What happens for $\theta > 10$?

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	