Sta 205 : Home Work #8

Due : March 29, 2006

I. **Convergence Concepts: a.s. and i.p.**

(A) Let $\{X_n\}$ be a monotonically increasing sequence of RVs such that $X_n \to X$ in probability (i.p). Show that $X_n \to X$ almost surely (a.s).

(B) Let $\{X_n\}$ be any sequence of RVs. Show that $X_n \to X$ a.s. if and only if

$$\sup_{k \geq n} |X_k - X| \to 0 \qquad \text{i.p.}$$

(C) Let $\{X_n\}$ be an arbitrary sequence of RVs and set $S_n = \sum_{i=1}^n X_i$. Show that $X_n \to 0$ a.s. implies that $S_n/n \to 0$ a.s.

(D) Let $\{X_n\}$, $\{Y_n\}$ be two sequences of RVs such that $0 \leq X_n \leq Y_n$ and $Y_n \to 0$ i.p. Show that $X_n \to 0$ i.p.

(E) Suppose $\{X_n\}$ are identically distributed with finite variance. Fix $\epsilon > 0$. Show that $n\mathbb{P}\left[|X_1| \geq \epsilon\sqrt{n}\right] \to 0$. Also show that $\frac{\bigvee_{i=1}^n |X_i|}{\sqrt{n}} \to 0$ i.p.

(F) For random variables X, Y define

$$\rho(X, Y) \equiv \inf\{\delta > 0 : \mathbb{P}[|X - y| \geq \delta] \leq \delta\}.$$

ρ defined above is a metric. Show that ρ "metrizes" convergence in probability: i.e, $X_n \to X$ i.p., if and only if $\rho(X_n, X) \to 0$.

II. L_p **Convergence**

(A) Let $\{X_n\}$ be a sequence of positive RVs such that $X_n \to X_0$ i.p. and $\mathbb{E}(X_n) \to \mathbb{E}(X_0)$. Show that $X_n \to X$ in L_1.

(B) For any two events A and B, define the distance $d(A, B)$ (Refer Hw 5, I F) as

$$d(A, B) \equiv \mathbb{P}(A \Delta B)$$

Prove that, for a sequence of events $\{A_n\}$, $d(A_n, A) \to 0$ if and only if $\mathbb{1}_{A_n} \to \mathbb{1}_A$ in L_2.

(C) Give an example of a sequence of RVs which converge in L_1 but do not converge in L_2. Let $\{X_n\}$ be a sequence of RVs such that $X_n \to X$ in L_1, $X \in L_2$. Give a simple condition on $\{X_n\}$ (which can be easily verified) such that $X_n \to X$ in L_2. Justify your answer.

(D) Let $([0,1], \mathcal{B}[0,1], \lambda)$ be our probability space, and define $X_n(\omega) \equiv \omega^n,\ \omega \in [0,1]$. For what $p \in [1, \infty]$, does the sequence $\{X_n\}$ converge in L_p ? If it does converge for some $p \in [1, \infty]$, find the limiting random variable as a function of the corresponding p. Explain your answer.

III. **Two Miscellaneous (But Interesting) Concepts**

(A) **Egoroff:**

i. Suppose $\{X_n\}, X$ are real valued RVs defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose for all $\omega \in \Lambda \in \mathcal{F}$, we have $X_n(\omega) \to X(\omega)$. Show that for every $\epsilon > 0$, there exists a set Λ_ϵ such that $\mathbb{P}(\Lambda_\epsilon) < \epsilon$ and

$$\sup_{\Lambda \setminus \Lambda_\epsilon} |X(\omega) - X_n(\omega)| \to 0 \qquad (n \to \infty)$$

Thus the convergence is uniform except on a small set. (For more on this problem see page 90)

ii. Now use Egoroff's theorem to show the following: Suppose $\{X_n\}$ be RVs such that $\sup_n |X_n| \le K < \infty$ a.s. and $X_n \to X$ a.s. Show that $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Notice that the last statement is an easy consequence of Dominated convergence theorem, but the point here is to use Egoroff!

(B) L_p **Spaces :**

i. For a random variable X, $1 < p < q < \infty$, show that

$$0 \le ||X||_1 \le ||X||_p \le ||X||_q \le ||X||_\infty$$

ii. For $1 < p < q < \infty$, show that

$$L_\infty \subset L_q \subset L_p \subset L_1$$

where L_p denotes the space of all RVs X with $||X||_q < \infty$. Hint: Jensen's inequality might be needed here.

iii. Show the following form of Holder's inequality : For RVs X, Y

$$\mathbb{E}(|XY|) \le ||X||_1 ||Y||_\infty$$

iv. Show the following form of Minkowski's inequality: For RVs X, Y

$$||X + Y||_\infty \le ||X||_\infty + ||Y||_\infty$$