Sta 205 : Home Work #7

Due : March 08, 2006

I. **True or False**? Answer whether each of the following statements is true or false. If your answer is true, answer why it is true. If it is false, show why— perhaps by giving a simple counter example.

(A) If $\{X_n,\ n \in \mathbb{N}\}$ is a uniformly integrable (U.I.) collection of random variables, then $X_n \in L_1$ for each n.

(B) Define a sequence $\{X_n\}$ of random variables on the unit interval with Lebesgue measure, $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, and $\mathsf{P} = \lambda$, by $X_n \equiv n \mathbb{1}_{(0, \frac{1}{n}]}$. Then $\{X_n\}$ is UI.

(C) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_2, *i.e.*, satisfies $\mathsf{E}|X_n|^2 \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

(D) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}|X_n| \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

II. **Characteristic Functions**.

(A) Let X be a random variable, and define

$$\phi_X(\omega) \equiv \mathbb{E}(e^{i\omega X}), \qquad \omega \in \mathbb{R}$$

Show that $\phi_X(\omega)$ is uniformly in continuous in $\mathbb{R}$.

(B) Find the characteristic functions of the following random variables :

i. $X \sim \mathsf{Ge}(p)$[1]

ii. $Y \sim \mathsf{Ex}(\lambda)$[2]

iii. $Z = X/n, \quad X \sim \mathsf{Ge}(\lambda/n)$

Find the limit of $\phi_Z(\omega)$ from part (iii) above as $n \to \infty$. Recognize it?

III. **Extra Credit : Infinite Divisibility.**
The distribution of a random variable X is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist n i.i.d random variables $\{Y_i\}$ such that X has the same distribution as $\sum_{i=1}^n Y_i$. Use characteristic functions to show that if $X \sim \text{Po}(\lambda)$, then X is infinite divisible. (Hint: Recall that random variables $\{Y_i\}$ are independent if and only if $\phi_{\Sigma Y_i}(\omega) = \prod \phi_{Y_i}(\omega)$ for all $\omega \in \mathbb{R}$) Which of the following distributions are infinitely divisible? Why?

$\mathsf{Un}(0, \theta)$? $\qquad$ $\mathsf{Bi}(n, p)$ for fixed n? $\qquad$ $\mathsf{Ga}(\alpha, \lambda)$ for fixed λ? $\qquad$ $\mathsf{Ge}(p)$?

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[1]Starting at $x = 0$— with p.m.f. $f(x \mid p) = p\, q^x$, $x = 0, 1, 2, ...$

[2]Rate parametrization— with p.d.f. $f(y) = \lambda\, e^{-\lambda\, y}$, $y > 0$.