Sta 205 : Home Work #6

Due : March 01, 2006

I. **Fubini and Tonelli**.

(A) Let X be a positive random variable (i.e, $X \geq 0$ $a.s$). Show that

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t)\ dt$$

Also verify that for any $\alpha > 0$

$$\mathbb{E}(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1}\mathbb{P}(X > t)dt$$

(B) Define $(\Omega_i, \mathcal{B}_i, \mu_i)$, for $i = 1, 2$ as follows: Let μ_1 be Lebesgue measure and μ_2 counting measure so that $\mu_2(A)$ is the number of elements of A. Let

$$\Omega_1 = (0,1),\ \mathcal{B}_1 = \text{ Borel sets of (0,1)}$$

$$\Omega_2 = (0,1),\ \mathcal{B}_2 = \text{ All subsets of (0,1)}$$

Define

$$f(x,y) = \mathbb{1}_{x=y}(x,y)$$

Let

$$I_1 = \int_{\Omega_1}\left[\int_{\Omega_2} f(x,y)\mu_2(dy)\right]\mu_1(dx)$$

$$I_2 = \int_{\Omega_2}\left[\int_{\Omega_1} f(x,y)\mu_1(dx)\right]\mu_2(dy)$$

Compute I_1 and I_2. Is $I_1 = I_2$? Are the measures μ_1 and μ_2 σ-finite? Why doesn't Fubini's theorem hold here?

(C) This problem is a probabilistic version of the familiar integration by parts from calculus. Suppose F and G are two distribution functions with no common points of discontinuity in an interval $(a,b]$. Show that

$$\int_{(a,b]} G(x)F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)$$

Show that the above formula need not hold true if F and G have common discontinuities.

II. **Uniform Integrability (UI)**.

(A) Let $\{X_n\}$ be a sequence of iid, L_1 random variables. Set $S_n \equiv \sum_{i=1}^{n} X_i$. Show that the sequence of random variables $\{Y_n\}$ defined by $Y_n \equiv S_n/n$ is UI.

(B) Let $X_n \sim \mathrm{N}(0, \sigma_n^2)$. Find a simple (easily verifiable) condition on $\{\sigma_n^2\}$ such that $\{X_n\}$ is UI.

(C) If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

(D) Suppose $\{X_n, n \geq 1\}$ is an **arbitrary** sequence of non-negative random variables, and set $M_n \equiv \vee_{i=1}^{n} X_i$. If $\{X_n\}$ is UI, show that $\mathbb{E}(M_n)/n \to 0$.

(E) Let $\phi(x)$ be a function which grows faster than x at infinity, i.e, $\phi(x)/x \to \infty$ as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that

$$\mathbb{E}(\phi(|Z|)) \leq B < \infty \qquad \forall Z \in \mathcal{C}$$

Show that $\mathcal{C}$ is UI.

III. **Convergence Theorems Revisited**.

(A) Let X be a non-negative real valued random variable. Show that:

i. $\lim_{n\to\infty} n\mathbb{E}(\frac{1}{X}\mathbb{1}_{[X>n]}) = 0$.

ii. $\lim_{n\to\infty} n^{-1}\mathbb{E}(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}) = 0$.

(B) Suppose $\{p_k, k \geq 0\}$ is a probability mass function on $\{0, 1, ...\}$ and define the generating function

$$P(s) = \sum_{k=0}^{\infty} p_k s^k \qquad 0 \leq s \leq 1$$

Prove using Dominated Convergence theorem that

$$\frac{d}{dx}P(s) = \sum_{k=1}^{\infty} p_k k s^{k-1} \qquad 0 \leq s \leq 1$$