Sta 205 : Home Work #5

Due : February 22, 2006

I. **Expectation**.

(A) Consider the triangle with vertices $(-1,0),(1,0),(0,1)$ and suppose (X_1, X_2) is a random vector uniformly distributed with in this triangle. Compute $\mathbb{E}(X_1 + X_2)$.

(B) Let $((0,1], \mathcal{B}((0,1]), \lambda)$ be a probability space (λ denotes the lebesgue measure). Let X be a random variable defined on the probability space described above, with $X(\omega) = 1,$ if $\omega \in \mathbb{Q}$ and 0 otherwise. What is $\mathbb{E}(X)$?

(C) Suppose $X \in L_1$. Show that

$$\int_{|X|>n} X d\mathbb{P} \to 0$$

as n tends to ∞.

(D) Let $\{A_n\}$ denote a sequence of events such that $\mathbb{P}(A_n) \to 0$ and let $X \in L_1$. Show that

$$\int_{A_n} X d\mathbb{P} \to 0$$

(E) Let $X \in L_1$, and let A be an event. Show that

$$\int_A |X| d\mathbb{P} = 0 \text{ iff } \mathbb{P}(A \cap [|X| > 0]) = 0$$

(F) Suppose that $(\Omega, \mathcal{B}, \mathbb{P})$, is a probability space and $A_i \in \mathcal{B},\ i \in \{1,2\}$. Define the distance $d : \mathcal{B} \times \mathcal{B} \mapsto \mathbb{R}$ by $d(A_1, A_2) \equiv \mathbb{P}(A_1 \Delta A_2)$. Let X be a random variable, $X \in L_1$. Show that, if $\{A_n\}, A \in \mathcal{B}$ such that $d(A_n, A) \to 0$, then

$$\int_{A_n} X d\mathbb{P} \to \int_A X d\mathbb{P}$$

Note: Here Δ denotes the symmetric set difference. For any two sets A and B, define $A \Delta B \equiv (A \setminus B) \cup (B \setminus A)$.

II. **Convergence Theorems**.

(A) Let X be a real valued random variable such that $X \geq 0$ almost surely. Set

$$S_n \equiv \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{P}\left(\frac{k}{2^n} < X \leq \frac{k+1}{2^n}\right) \qquad n \in \mathbb{N}$$

Note that S_n is a sequence of real numbers. What is $\lim_{n\to\infty} S_n$? Justify your answer.

(B) Let $\{X_n\}$ be a sequence of random variables defined on the probability space $((0,1], \mathcal{B}((0,1]), \lambda)$ such that

$$X_n = \frac{n}{\log n} \mathbb{1}_{(0,\frac{1}{n})} \qquad n \in \mathbb{N}$$

Show that $X_n \to 0$ almost surely, and $\mathbb{E}(X_n) \to 0$. However, verify that here the conditions required for the dominated convergence theorem to hold fails.

(C) Suppose $\{Y_n\}$ be a sequence of random variables such that

$$\mathbb{P}(Y_n = \pm n^3) = \frac{1}{2n^2}, \qquad \mathbb{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

Show using the Borel-Cantelli lemma that $Y_n \to 0$ almost surely. Compute $\lim_{n\to\infty} \mathbb{E}(Y_n)$. It is 0? Is the Lebesgue dominated convergence theorem applicable? Why or why not?

(D) Let $\{X_n\}, X$ be random variables, and $0 \leq X_n \to X$. If $\sup_n \mathbb{E}(X_n) = K(\text{a constant}) < \infty$, then show that $\mathbb{E}(X) \leq K$ and $X \in L_1$. Also show that $\mathbb{E}(X_n) \to \mathbb{E}(X)$.

III. **Potpourri**.

(A) If X, Y are independent random variables and $\mathbb{E}(X)$ exists, then for all $B \in \mathcal{B}(\mathbb{R})$, show that

$$\int_{[Y\in B]} X dP = \mathbb{E}(X)\mathbb{P}[Y \in B]$$

(B) Let $\{X_n\}$ be a sequence of (not necessarily independent) Bernoulli random variables with

$$\mathbb{P}(X_n = 1) = p_n = 1 - \mathbb{P}(X_n = 0)$$

Show that $\sum_{n=1}^{\infty} p_n < \infty$ implies $\sum_{n=1}^{\infty} \mathbb{E}(X_n) < \infty$ and hence conclude that $X_n \to 0$ almost surely.

(C) Let $\{X_n\}$ be a sequence of random variables. Show that

$$\mathbb{E}(\bigvee_{n=1}^{\infty} |X_n|) < \infty$$

if and only if there exists a random variable $0 \leq Y \in L_1$ such that

$$\mathbb{P}(|X_n| \leq Y) = 1, \qquad \forall n \geq 1$$

Here $\bigvee |X_n|$ denotes the supremum.