Sta 205 : Home Work #4

Due : February 15, 2006

I. **Indepedence**.

(A) Let $B_1, B_2, ..., B_n$, be independent events. Show that

$$\mathbb{P}(\bigcup_{i=1}^{N} B_i) = 1 - \prod_{i=1}^{N}(1 - \mathbb{P}(B_i))$$

(B) If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that

$$\mathbb{P}(A_n A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m) \ \forall n, m \in \mathbb{N}$$

Does it follow that the events $\{A_n\}$ independent? Give a proof or counter example.

(C) Let X be a random variable. Show that X is independent of itself if and only if it X is a constant with probability 1. Let f be a Borel measurable function. Can $f(X)$ and X be independent? Explain your answer.

(D) Show that if the event A is independent of the π-system $\mathcal{P}$ and $A \in \sigma(\mathcal{P})$, then $\mathbb{P}(A)$ is either 0 or 1.

(E) Give a simple example to show that two random variables may be independent according to one probability measure dependent with respect to another.

II. **Practice with Borel Cantelli**.

(A) Let $\{X_n\}$ be a sequence of Bernoulli (binary) random variables, such that $\mathbb{P}(X_n = 1) = \frac{1}{n^2}$. Now consider the sum $S_n = \sum_{k=1}^{n} X_k$. Show that S_n converges almost surely. Now let's assume that $\{X_n\}$ are indepedent. Does S_n converge, if $\mathbb{P}\big(X_n = 1\big) = \frac{1}{n}$? Why or why not?

(B) Dane is tossing a **heavily biased** coin infinitely many times, and outcomes the tosses are independent. He is convinced that if he chooses the probability of heads to be very small (say ≈ 0.05), then he will get heads only finitely many times with probability 1. Is Dane right? Justify your answer.

(C) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Let $\{X_n\}$ be a sequence of **iid** non trivial (i.e., not almost-surely constant) random variables , then show that

$$\mathbb{P}[X_n \text{ converges}] = 0$$

(D) Use the Borel-Cantelli lemma to prove that for any sequence of real valued random variables $\{X_n\}$, there exists constants $c_n \to \infty$ such that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Give a careful description of how you choose c_n. Let $\{X_n\}$ be iid standard Gaussian random variables. Give an example of such a sequence $\{c_n\}$ such that $X_n/c_n \to 0$ almost surely.

III. **Mixed Bag**.

(A) Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathbb{P}(A_n) < 1, \forall n \in \mathbb{N}$. Show that $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathbb{P}(A_n \text{ i.o.}) = 1$. Give an example to show that the condition $\mathbb{P}(A_n) < 1$ cannot be dropped.

(B) Suppose $\{A_n\}$ is a sequence of events. If $\mathbb{P}(A_n) \to 1$ as $n \to \infty$, prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathbb{P}(\cap_k A_{n_k}) > 0$.

(C) Let A_n be a sequence of events. If there exists $\epsilon > 0$ such that $\mathbb{P}(A_n) \geq \epsilon$ for all $n \in \mathbb{N}$, does it follow that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathbb{P}(\cap_k A_{n_k}) > 0$? Why or why not?

(D) Let $\{X_n\}$ be iid random variables, with tail σ-field

$$\mathcal{T} \equiv \cap_n \mathcal{F}_n, \qquad \mathcal{F}_n \equiv \sigma\{X_m : m \geq n\}$$

Is the event

$$E = \{\text{There exists a number } B < \infty \text{ such that } |X_n| \leq B \text{ for infinitely-many } n\}$$

in $\mathcal{T}$? Prove it. Find the probability $\mathbb{P}[E]$ in terms of the random variables' common CDF, $F(x) \equiv \mathbb{P}[X_n \leq x]$.