Sta 205 : Home Work #3

Due : February 08, 2006

I. **Random variables**.

(A) Let $(\Omega, \mathcal{B}, \mathbb{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$ where λ is Lebesgue measure. Define

$$\begin{aligned} X_1(\omega) &\equiv 0, \ \forall \omega \in \Omega \\ X_2(\omega) &\equiv \mathbb{1}_{\{1/2\}}(\omega) \\ X_3(\omega) &\equiv \mathbb{1}_{\mathbb{Q}}(\omega) \end{aligned}$$

where $\mathbb{Q}$ is set of rational numbers in $(0,1]$. Let $f(k) \equiv \mathbb{P}[X_1 = X_2 = X_3 = k]$, $k \in \mathbb{R}$. Plot the function $f(\cdot)$. What are the σ - algebra's generated by X_1, X_2 and X_3?

(B) Let's toss two coins and define a random variable ζ as follows: If both the coins land on heads, $\zeta = 1$, if both the coins land on tails $\zeta = 2$, and if the outcome happens to be a head and a tail, $\zeta = 0$. Our goal is to formalize the above set-up. What is a suitable sample space (Ω) here? What is the σ-algebra $\mathcal{F}$ generated by ζ? Give a probability assignment on $\mathcal{F}$ assuming that the coins are fair and the outcomes of the two tosses are independent of each other. Let $\mathcal{P}$ denote the power set of Ω. Is $\mathcal{F}$ equal to $\mathcal{P}$? Is ζ measurable with respect to $\mathcal{P}$?

II. **More on Random variables**.

(A) Let X be a random variable and let $F(x) = \mathbb{P}(X \leq x)$. Set $Y \equiv F(X)$. If X has a continuous distribution, show that Y is a random variable and that Y has a uniform distribution on $[0,1]$.

(B) If X is a real valued random variable (so $\mathbb{P}[|X| < \infty] = 1$), then show that for any $\epsilon > 0$, there exists a *bounded* random variable Y such that

$$\mathbb{P}(X \neq Y) < \epsilon$$

(A random variable Y is bounded if $\exists$ some $K_\epsilon < \infty$ (K_ϵ does not depend on ω), such that $\forall \omega \in \Omega$, $|Y(\omega)| \leq K_\epsilon$.)

III. **Measurable functions**.

(A) Let $\Omega = \mathbb{R}$, and set $\mathcal{S} = \{\Omega, \emptyset, (-\infty, 0], (0, \infty)\}$. Show that $\mathcal{S}$ is a σ- algebra. What functions f defined on Ω are $\mathcal{S}$ measurable ?

(B) If X is a random variable, then show that so is $|X|$. Show by an example that the converse need not be true.

(C) Let $\Omega = \mathbb{R}$. Let $\mathcal{S}_0 \equiv \{\emptyset, \Omega\}$. $\mathcal{S}_0$ is called the trivial σ-algebra. Consider the function $f(x) = x^2$. Is the function f measurable with respect to $\mathcal{S}_0$? Justify your answer. Give two other σ-fields $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{B}(\mathbb{R})$, and f is measurable with respect to $\mathcal{S}_2$ and f is **not** measurable with respect to $\mathcal{S}_1$. Explain your answer.

(D) Suppose $\{X_n, n \geq 1\}$ are random variables on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and define the induced "random walk" by

$$S_0 = 0, \; S_n \equiv \sum_{i=1}^{n} X_i, \; n \geq 1$$

Let

$$\tau \equiv \inf\{n > 0 : S_n > 0\}$$

Prove that τ is a random variable. **Extra credit**: Assume that we also know that $\tau(\omega) < \infty, \forall \omega \in \Omega$. Prove that S_τ is also a random variable.

IV. **Practice with limsup and liminf**.

(A) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on Ω and set $Y \equiv \limsup X_n$. For $\beta \in \mathbb{R}$, express the event $\{\omega \in \Omega : Y(\omega) \geq \beta\}$ in terms of unions and interesections of the events of the form $\{\omega \in \Omega : X_n(\omega) \geq \beta\}$. Why is Y a random variable ?

(B) Now set $Z \equiv \liminf X_n$. For $\alpha \in \mathbb{R}$, express the event $\{\omega \in \Omega : Z(\omega) \geq \alpha\}$ in terms of unions and interesections of the events of the form $\{\omega \in \Omega : X_n(\omega) \geq \alpha\}$. Why is Z a random variable ?