

Sta 205 : Home Work 2

Due : February 01, 2006

I. σ - algebra and probability assignments.

- (A) Let A, B, C be three events in a probability space. Enumerate all the elements of $\sigma(A, B, C)$, the sigma algebra generated by the above events.
- (B) Let's further assume that the above mentioned events A, B, C are disjoint, and $\mathbb{P}(A) = 0.6, \mathbb{P}(B) = 0.3, \mathbb{P}(C) = 0.1$. Calculate the probability of every event in $\sigma(A, B, C)$.

II. Practice with null sets.

- (A) Let $\{A_n, n \in \mathbb{N}\}$ be events such that $\mathbb{P}(A_n) = 0, \forall n$. Show that $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = 0$.
- (B) Let $\{B_n, n \in \mathbb{N}\}$ be events such that $\mathbb{P}(B_n) = 1, \forall n \in \mathbb{N}$. What is $\mathbb{P}(\cap_{n=1}^{\infty} B_n)$?
- (C) Now consider the set of events, $\{E_\alpha, \alpha \in \mathbb{R}\}$, such that $\mathbb{P}(E_\alpha) = 0, \forall \alpha \in \mathbb{R}$. Does it necessarily follow that $\mathbb{P}(\cup_{\alpha \in \mathbb{R}} E_\alpha) = 0$? If yes, give a proof, otherwise give a counter example.
- (D) Finally, let $\{B_k\}$ be a collection of events such that, $\sum_{k=1}^n \mathbb{P}(B_k) > n - 1$. Show that $\mathbb{P}(\cap_{k=1}^n B_k) > 0$.

III. Distribution functions and continuity.

- (A) Give an example of a function which is continuous on \mathbb{R} , but **not** uniformly continuous.
- (B) Let G be a continuous distribution function on \mathbb{R} . Show that G is in fact uniformly continuous.
- (C) Now let F be any distribution function on \mathbb{R} . Show that F can have **at most countably** many discontinuities.
- (D) Let F again denote any distribution function. Define

$$F_l^{\leftarrow}(y) = \inf\{t : F(t) \geq y\}$$

$$F_r^{\leftarrow}(y) = \inf\{t : F(t) > y\}$$

Show that, $\lambda\{u \in (0, 1] : F_l^{\leftarrow}(u) \neq F_r^{\leftarrow}(u)\} = 0$, where $\lambda(\cdot)$, denotes the lebesgue measure.

IV. π & λ - systems.

(A) Let $\Omega = [0, 1] \times [0, 1]$, and consider the following collections of subsets of Ω :

$$\mathcal{A} = \{[0, a] \times [0, b] : 0 < a, b < 1\}$$

- i. Is \mathcal{A} a π - system? Why or why not?
- ii. Is \mathcal{A} a λ - system? Why or why not?

(B) Consider the following collection of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b], b \in \mathbb{R}\}$$

- i. Show that \mathcal{B} is a π - system, but not a λ system.
- ii. What is the λ - system generated by \mathcal{B} ?

V. π - systems and fields.

(A) Let \mathcal{C} be a non empty collection of subsets of Ω , and let $\mathcal{A}(\mathcal{C})$ be the minimal field over \mathcal{C} . Show that $\mathcal{A}(\mathcal{C})$ consists of sets of the form

$$\cup_{i=1}^m \cap_{j=1}^{n_i} A_{ij},$$

where for each i, j either $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and where the m sets $\cap_{j=1}^{n_i} A_{ij}$, $1 \leq i \leq m$, are disjoint. Thus, we can explicitly represent the sets in $\mathcal{A}(\mathcal{C})$, however it turns out that, we cannot do the same for the σ -field over \mathcal{C} .

(B) Now let's further assume that \mathcal{C} is a π system. Show that if $\mathbb{P}_1, \mathbb{P}_2$ are two probability measures which agree on \mathcal{C} , then $\mathbb{P}_1, \mathbb{P}_2$ must also agree on $\mathcal{A}(\mathcal{C})$. Hint: Use part(A) and the inclusion-exclusion principle.