Midterm Examination #2

STA 205: Probability and Measure Theory

Wednesday, 2004 Mar 24, 2:20-3:35 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. You may use a calculator but not a laptop, pda, etc. If a question seems ambiguous or confusing *please* ask Jason or Zhenglei— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Print Name: ______________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\Omega = \mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. The sum $\zeta(p) \equiv \Sigma_{n=1}^{\infty} \frac{1}{n^p}$ is infinite for $p \le 1$ but finite for all $p > 1$; for $p = 4$ it is $\zeta(4) = \Sigma_{n=1}^{\infty}(1/n^4) = \pi^4/90$, so the set-function

$$\mathsf{P}[A] \equiv \frac{90}{\pi^4} \sum_{n \in A} \frac{1}{n^4}$$

is a probability assignment on the power set $\mathcal{F} = 2^{\Omega}$ of all subsets of Ω. Define a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ by $X(\omega) \equiv \omega$.

a. (5) For $p > 0$, is $X \in L^p(\Omega, \mathcal{F}, \mathsf{P})$? If this depends on p, tell which L^p spaces contain X. Choose one and give your reasoning:

◯ $X \in L^p$ for no $0 < p < \infty$ ◯ $X \in L^p$ for ____ $< p <$ ____

b. (5) Define a sequence of sequence of truncated approximations to X by $X_n(\omega) \equiv \min(n, \omega)$. Are *they* in L^p? Why?

◯ $X_n \in L^p$ for no $0 < p < \infty$ ◯ $X_n \in L^p$ for ____ $< p <$ ____

c. (5) Does $X_n \to X$ in probability? ◯ Yes ◯ No Explain:

d. (5) Does $X_n \to X$ in L^1? Pick one: ◯ Yes ◯ No Explain:

Problem 2: Let X_n be independent random variables, all uniformly distributed on the interval $(0, \theta]$ for some $\theta > 0$. Let $X_n^* \equiv \max_{1 \le i \le n} X_i$ be the maximum of the first n X_i's. In which of the following senses (if any) does $X_n^* \to \theta$ as $n \to \infty$? Mark each sense in which convergence takes place, and show (by calculation) that you are correct:

○ *a.s* ○ L^∞ ○ L^2 ○ L^1 ○ *in pr.*

Explanation:

Problem 3: Answer the following questions about a sequence $\{U_n\}$ $(n = 1, 2, ...)$ of independent random variables, all uniformly distributed on the interval $(0, 1]$:

a. (5) What is the probability that infinitely-many of the events

$$A_n = \{\omega : U_n(\omega) < \frac{1}{n}\}$$

occur?

b. (5) What is the probability that *all* of the events

$$B_n = \{\omega : U_n(\omega) \leq \exp\big(-2^{-n}\big)\}$$

occur?

c. (5) What is the probability that infinitely-many of the events

$$C_n = \{\omega : U_1(\omega) < U_2(\omega) < ... < U_n(\omega)\}$$

occur?

d. (5) Does the sequence of random variables $X_n \equiv (U_n)^n$ converge to zero in probability? ◯ Yes ◯ No Why?

Problem 4: For two sequences of real numbers $a_n \in \mathbb{R}$ and $b_n \in (0,1]$ and a sequence $\{U_n\}$ of independent random variables, each uniformly distributed on $(0,1]$, define random variables by

$$X_n = a_n \, 1_{(0,b_n]}(U_n) = \begin{cases} a_n & 0 < U_n \le b_n \\ 0 & b_n < U_n \le 1 \end{cases}$$

The random variables $\{X_n\}$ converge to zero in L^1 if and only if the sequences $\{a_n\}$ and $\{b_n\}$ satisfy the condition $|a_n| b_n \to 0$.

a. (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ in L^2? Why?

b. (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ in L^∞? Why?

c. (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ *in pr.*? Why?

d. (5) What condition must $\{a_n\}$ and $\{b_n\}$ satisfy for $X_n \to 0$ *a.s.*? Why?

Problem 5: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky (no tricks are intended).

a. T F Lebesgue's monotone convergence theorem implies that $\int_0^1 \sin(n\pi x)\,dx \to 0$.

b. T F Jensen's Inequality implies that $\mathsf{E}e^X \geq e^{\mathsf{E}X}$ for any $X \in L^1$.

c. T F For any r.v. $Y \geq 0$ and number $a > 0$, $\mathsf{P}[Y > a] \leq \mathsf{E}Y/a$.

d. T F For any sequence of positive (measurable) functions $f_n(x) \geq 0$ on $\mathbb{R}$, then

$$\int \left[\sum_n f_n(x)\right] dx = \sum_n \left[\int f_n(x)\,dx\right].$$

e. T F If $X_n \to X$ *in pr.* then Lebesgue's dominated convergence theorem implies that $\mathsf{E}e^{itX_n}$ converges to $\mathsf{E}e^{itX}$ at every $t \in \mathbb{R}$.

f. T F Let X have the geometric distribution with $\mathsf{P}[X = k] = 2^{-k}$ for $k = 1, 2, ...$ Then $\mathsf{P}[X \text{ is even}] < 1/2$.

g. T F Two σ-fields $\mathcal{F}$, $\mathcal{G}$ are independent if and only if $\mathsf{P}[F \cap G] = \mathsf{P}[F]\mathsf{P}[G]$ for every $F \in \mathcal{F}$, $G \in \mathcal{G}$.

h. T F Two random variables X and Y are independent if and only if $\mathsf{E}[f(X \cdot Y)] = E[f(X)]\,E[f(Y)]$ for every Borel function $f(x)$.

i. T F If $X \in L^1$ then $X \in L^2$ and $\mathsf{E}[X^2] \leq \mathsf{E}[\,|X|\,]^2$.

j. T F If each X_n has a discrete distribution taking only rational values and if X_n converges in distribution to X then X is discrete too, since $\mathsf{E}g(X_n) \to \mathsf{E}g(X)$ where $g(x)$ is the indicator of the rationals.

Blank Worksheet

Another Blank Worksheet