Midterm Examination

STA 205: Probability and Measure Theory

Thursday, 2003 Feb 27, 12:40-1:55 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. You may use a calculator but not a laptop, pda, etc. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Print Name: ________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the probability space $\Omega = \{a, b, c, d\}$ with σ-field $\mathcal{F} = 2^\Omega$ and probability assignment given by $\mathsf{P}(\{\omega\}) = \{0.2, 0.1, 0.3, 0.4\}$ for $\omega = \{a, b, c, d\}$, respectively.

a. (5) List *all* possible values $0 \le p \le 1$ for which it is possible to construct a random variable X_p on $(\Omega, \mathcal{F}, \mathsf{P})$ with the Bernoulli distribution with $\mathsf{P}[X_p = 1] = p = 1 - P[X = 0]$. If possible, give X_p explicitly for $p = 1/2$; if not, tell why not.

$p \in \{$ ________________ $\}$ $\qquad$ $X_{1/2}(\omega) =$ ________________

b. (5) Is it possible to construct a random variable Y on $(\Omega, \mathcal{F}, \mathsf{P})$ with exactly three possible outcomes, all equally likely? If so, do it (explicitly); if not, tell why.

◯ Yes ◯ No $\qquad Y(\omega) =$ ________________

Problem 1 (cont'd): Recall $\Omega = \{a, b, c, d\}$ and $\mathsf{P}(\{\omega\}) = \{0.2, 0.1, 0.3, 0.4\}$ for $\omega = \{a, b, c, d\}$.

c. (5) Give (explicitly) the σ-field $\mathcal{G} = \sigma(W, Z)$ generated by the two random variables $W(\omega) = 1_{\{c\}}(\omega)$ and $Z(\omega) = 1_{\{a,b\}}(\omega)$ (where $1_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise).

$\mathcal{G} =$ ______________

d. (5) Is your σ-field $\mathcal{G}$ (from c. above) *complete* for P?
◯ Yes ◯ No Why?

Problem 2: A real-valued function ϕ on a convex set S is *convex* if the line segment connecting any two points $[x, \phi(x)]$ and $[y, \phi(y)]$ on the graph $\{[s, \phi(s)] : s \in S\}$ of ϕ lies above the graph:

$$\forall x, y \in S,\ 0 < t < 1, \quad \phi\big(x + t(y - x)\big) \le \phi(x) + t\big[\phi(y) - \phi(x)\big]$$

If ϕ is continuous it is enough to check this for the midpoint, i.e., a continuous function ϕ is convex if

$$\forall x, y \in S, \quad \phi\left(\frac{x+y}{2}\right) \le \frac{\phi(x) + \phi(y)}{2}.$$

A twice differentiable function ϕ on an interval is convex if $\phi''(x) \ge 0$ everywhere. Let ϕ, ψ both be continuous and convex on an interval $S \subset \mathbb{R}$. You may assume differentiability if you wish. Choose Yes or No (3 pts) and give support each answer with a brief argument or a sketch of the graphs of ϕ and ψ (2pts):

a. (5) Is the sum $(\phi + \psi)$ convex? ◯ Yes ◯ No

b. (5) Is the difference $(\phi - \psi)$ convex? ◯ Yes ◯ No

c. (5) Is the maximum $(\phi \vee \psi)$ convex? ◯ Yes ◯ No

d. (5) Is the minimum $(\phi \wedge \psi)$ convex? ◯ Yes ◯ No

Problem 3: Some economists use the *Pareto* distribution to model incomes; a simple version of this distribution has

$$\mathsf{P}[X > x] = x^{-\alpha}, \qquad x > 1$$

for some parameter $\alpha > 0$. Let X have this Pareto distribution.

a. (5) Give the probability density function $f(x)$ for X. Be careful about the support, *i.e.*, the set where $f(x) > 0$.

$f(x) =$ ______________

b. (15) For which real numbers $p \in (0, \infty)$ is $X \in L_p$? Note this may depends upon α. Evaluate $\mathsf{E}[X^p]$.

$X \in L_p$ if $p \in \{$ ________________________ $\}$

$\mathsf{E}[X^p] =$ ______________

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with Lebesgue measure $\mathsf{P} = \lambda$; for each number $\alpha \in \mathbb{R} = (-\infty, \infty)$, consider the sequence of random variables (for $n \in \mathbb{N} = \{1, 2, ...\}$)

$$X_n(\omega) \equiv n^\alpha 1_{(0,1/n]}(\omega) = \begin{cases} n^\alpha & 0 < \omega \le 1/n \\ 0 & \text{otherwise.} \end{cases}$$

Answer each question below and show why your answer is correct.

a. (4) For each $\alpha \in \mathbb{R}$ compute the probability of $\{\omega : X_n(\omega) \to 0\}$:

$\mathsf{P}[X_n \to 0] =$ ______________

b. (8) For each $\alpha \in \mathbb{R}$ and $p > 0$ compute $\mathsf{E}\big[|X_n|^p\big]$. For which $\alpha \in \mathbb{R}$ and $p > 0$ does $X_n \to 0$ in L_p?

$\mathsf{E}\big[|X_n|^p\big] =$ ______________ $\mathsf{E}\big[|X_n|^p\big] \to 0$ if $p \in$ ______________

c. (8) For which $\alpha \in \mathbb{R}$ (if any) is $\{X_n : n \in \mathbb{N}\}$ "dominated" by some $Y \in L_1((\Omega, \mathcal{F}, \mathsf{P}))$ satisfying $|X_n| \le Y$? Find such a Y if possible.

$\alpha \in \{$ ____________________ $\}$ $Y =$ ______________

Problem 5: Let X be a simple random variable (with finitely-many values) and Y a continuously-distributed random variable on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $\mathsf{P} = \lambda$. In each case give an explicit non-trivial example of X and Y if possible; if impossible, say why. Do not prove anything!

a. (4) Is it possible that $\sigma(X) \subset \sigma(Y)$? ◯ Yes ◯ No

b. (4) Is it possible that $\sigma(Y) \subset \sigma(X)$? ◯ Yes ◯ No

c. (4) Is it possible that $\sigma(X) \perp\!\!\!\perp \sigma(Y)$? ◯ Yes ◯ No

d. (4) Is it possible that $\sigma(X) = \mathcal{F}$? ◯ Yes ◯ No

e. (4) Is it possible that $\sigma(Y) = \mathcal{F}$? ◯ Yes ◯ No

Blank Worksheet

Another Blank Worksheet