Final Examination

STA 205: Probability and Measure Theory

Due Monday, 2002 Apr 29, 5:00 pm

This is an open-book take-home examination. You must do your own work— collaboration is not permitted. If a questions seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 5pm Monday, 2002 April 29. You can slip them under my office door (211c Old Chem).

Print Name: ______________________

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Problem 1: Let $\xi_n \sim \mathsf{Ge}(1/2)$ be independent geometric-distributed random variables with probability mass function (pmf)

$$\mathsf{P}[\xi_n = x] = \begin{cases} 2^{-1-x} & x = 0, 1, ... \\ 0 & \text{other } x. \end{cases}$$

and set

$$X_n = 1_{[0,n)}(\xi_n) = \begin{cases} 0 & \xi_n < n \\ 1 & \xi_n \geq n. \end{cases}$$

a. (5) Find the mean $\mu_n = \mathsf{E}[X_n]$ and variance $\sigma_n^2 = \mathsf{Var}[X_n]$ of X_n (*not* of ξ_n):

$\mu_n =$ ______________ $\sigma_n^2 =$ ______________

b. (5) Set $S_n = X_1 + ... + X_n$. Does S_n converge as $n \to \infty$? In what sense? Why?

c. (5) Find the characteristic function of ξ_n:

$\phi(\omega) = \mathsf{E}\left[e^{i\omega\xi_n}\right] =$ ______________

d. (5) Find the characteristic function of X_n:

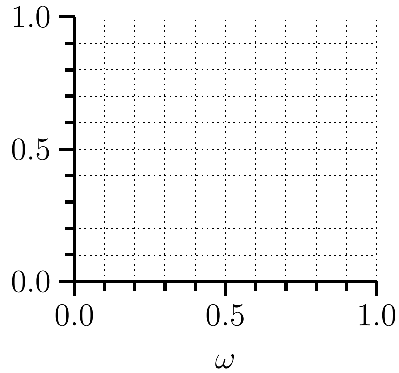
$\phi_n(\omega) = \mathsf{E}\left[e^{i\omega X_n}\right] =$ ______________

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}_1, d\omega\big)$ be the unit interval with Lebesgue measure (length). Define three random variables by

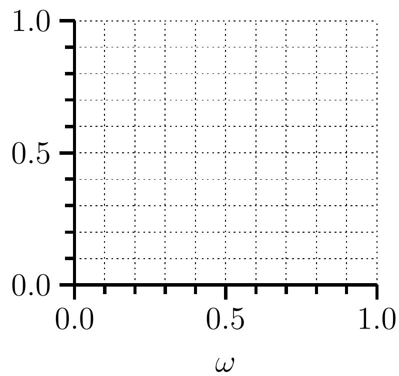
$$X(\omega) = 1_{(0,.5]}(\omega) \qquad Y(\omega) = 1_{(.3,1]}(\omega) \qquad Z(\omega) = \omega.$$

a. (10) What is the σ-algebra $\sigma(X,Y)$ generated by X and Y? How many events does it include? ____ What are they?

b. (5) Find and plot the conditional expectation $\mathsf{E}[Z|X,Y]$:



c. (5) Find and plot the conditional expectation $\mathsf{E}[W|X,Y,Z]$ for $W(\omega) \equiv \omega^2$ (Hint: No integration will be necessary):



Problem 3: Some economists use the *Pareto* distribution to model incomes; a simple version of this distribution has

$$\mathsf{P}[X > x] = x^{-\alpha}, \qquad x > 1$$

for some parameter $\alpha > 0$. Let $X_1,\ X_2, ...$ be independent random variables from this distribution and set $S_n \equiv X_1 + ... + X_n$.

a. (5) Give the probability density function $f(x)$ for X_1. Be careful about the support (*i.e.*, the set where $f(x) > 0$).

b. (5) For which $p > 0$ is $X_1 \in L^p$? If this depends on p and/or α, tell which L^p spaces contain X_1 for which α and evaluate $\mathsf{E}[{X_1}^p]$.

Problem 3 (continued):

c. (5) For which α (if any) does S_n/n converge as $n \to \infty$, and in what sense does it converge? Why? Give the limit where possible.

d. (5) For which α do there exist sequences a_n, b_n of real numbers such that $(S_n - a_n)/b_n \Rightarrow Z$ converges in distribution as $n \to \infty$? Give the sequences $a_n,\ b_n$ and the limit where possible.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with Lebesgue measure; for each number $\alpha \in \mathbb{R}$, consider the sequence of random variables (for $n \in \mathbb{N} = \{0, 1, 2, ...\}$)

$$X_n \equiv 2^{\alpha n} 1_{(0,2^{-n}]}(\omega) = \begin{cases} 2^{\alpha n} & 0 < \omega \leq 2^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Answer each question below and show why your answer is correct.

a. (5) For which α (if any) does $X_n \to 0$ almost surely?

b. (5) Fix $p \geq 1$. For which α (if any) does $X_n \to 0$ in L^p?

c. (5) For which α (if any) is $\{X_n : n \in \mathbb{N}\}$ "dominated" by some $Y \in L^1(\Omega, \mathcal{F}, \mathsf{P})$ satisfying $|X_n| \leq Y$? Find such a Y and verify its integrability where possible.

d. (5) For which α (if any) is $\{X_n : n \in \mathbb{N}\}$ uniformly integrable? Why?

Problem 5: The random variables $\{X_i\}$ are all independent and all satisfy $\mathsf{E}[{X_i}^4] \le 1.0$, but they may have different distributions. Let $S_n \equiv \sum_{i=1}^n X_i$ be their partial sum.

a. (10) Does it follow without any further assumptions that S_n/n converges almost surely? Give a proof or counter-example.

b. (10) If in addition we know $\mathsf{E}[X_i] = 0$ and $\mathsf{E}[{X_i}^2] = \sigma^2$ for all i, does it follow without further assumptions that $S_n/\sqrt{n}$ converges in distribution? Give (or cite) a proof or counter-example.

Problem 6: A sequence of integrable functions $f_n \in L^1(\mathcal{U}, \mathcal{B}, dx)$ on the unit interval $\mathcal{U} = (0, 1]$ satisfies

$$\lim_{n\to\infty} f_n(x) = f(x)$$

at every point $x \in \mathcal{U}$, for some limit function $f \in L^1(\mathcal{U}, \mathcal{B}, dx)$.

a. (10) Does it follow without any further assumptions that $\int_{\mathcal{U}} f_n(x)\,dx$ converges to $\int_{\mathcal{U}} f(x)\,dx$? Give a proof or a counter-example.

b. (10) If in addition we know that $f_n(x) \geq 0$ and $\int_{\mathcal{U}} f_n(x)\,dx = \int_{\mathcal{U}} f(x)\,dx = 1$, so that $f(x)$ and each $f_n(x)$ is a probability density function, and if we still have $f_n(x) \to f(x)$ for all x, does it follow that random variables $X_n \sim f_n(x)\,dx$ with these distributions converge in distribution to a limit $X \sim f(x)$? Why?