

Probability Midterm Take-Home

Assigned: 1:55pm Thursday, November 5, 1998

Due: 12:40pm Tuesday, November 10, 1998

Please try to answer each question. You may use your notes, the text, or any other books. I invite you to ask me questions, but please do not talk to each other about the questions.

You must **show** some **work** to get credit— *unsupported answers are not acceptable*. Attach any necessary work sheets to the assignment before returning it; be sure to put your name on each page. It is to your advantage to write your solutions as clearly as possible.

Probability Midterm Take-Home

1. Let X and Y be independent and identically-distributed random variables on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, each taking on the values $\{1, 2, 3, 4, 5, 6\}$ with probabilities $2/8, 1/8, 1/8, 1/8, 1/8, 2/8$, respectively; these are intended to represent the numbers showing on two “loaded” dice, weighted to make ones and sixes come up a little more often than usual.

a. Construct a suitable probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and represent X and Y as \mathcal{F} -measurable functions on Ω . Be sure to specify \mathbf{P} completely, by describing how to calculate $\mathbf{P}[A]$ for *every* $A \in \mathcal{F}$. You need not prove measurability.

b. Find the smallest σ -field $\sigma(S)$ for which the sum $S = X + Y$ is measurable. Is X $\sigma(S)$ -measurable?

c. Find the probability $\mathbf{P}[S = 7]$. Is it bigger, smaller, or the same as the probability of the event $[X + Y = 7]$ for independent *fair* dice?

d. Are the events $[X \text{ is divisible by } 2]$ and $[X \text{ is divisible by } 3]$ independent under \mathbf{P} ? Show why or why not.

Probability Midterm Take-Home

2. Let \mathcal{F}_n be the σ -field on $\Omega = (0, 1]$ generated by sets of the form $(\frac{i}{2^n}, \frac{j}{2^n}]$ for $0 \leq i < j \leq 2^n$, and let $\mathcal{B} = \sigma(\cup_n \mathcal{F}_n)$ be the Borel σ -field they generate. Let \mathbf{P} denote Lebesgue measure on each of these σ -fields.

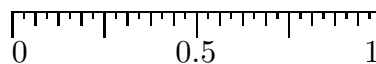
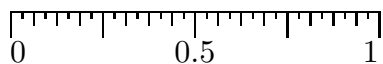
a. Prove that the random variable $X(\omega) = \omega^2$ is *not* \mathcal{F}_n -measurable for any $n < \infty$.

b. For each n there is a random variable X_n which is \mathcal{F}_n -measurable and which approximates X in the sense that $\mathbf{E}X_n Y = \mathbf{E}XY$ for every bounded \mathcal{F}_n -measurable Y or, equivalently, that:

$$\int_E (X_n - X) d\mathbf{P} = 0 \quad (\star)$$

for every $E \in \mathcal{F}_n$. Find X_n for $n = 1$ explicitly. Does (\star) determine each X_n uniquely? Why?

c. Sketch the graphs of $X_1(\omega)$ and $X_2(\omega)$ as functions of ω . Label the heights.



d. Does (\star) hold for *every* Borel set $E \in \mathcal{B}$? Prove it.

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3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the unit interval with Lebesgue measure; for each real number α , $-\infty < \alpha < \infty$, consider the sequence of random variables

$$X_n(\omega) = n^\alpha 1_{(0, 1/n]}(\omega),$$

equal to n^α if $0 < \omega \leq 1/n$ and zero otherwise. Answer each question below; show why your answer is correct.

(a.) For which α (if any) does $X_n \rightarrow 0$ *in pr.*?

(b.) For which α (if any) does $X_n \rightarrow 0$ *a.s.*?

(c.) For which α (if any) does $X_n \Rightarrow 0$ *in dist.*?

(d.) Fix a real number $p > 0$. For which α (if any) does $X_n \rightarrow 0$ in L^p ?

(e.) For which α (if any) does $X_n \rightarrow 0$ in L^∞ (*i.e.*, uniformly)?

(f.) For which α (if any) are $\{X_n\}$ uniformly integrable?

