## PRIOR TO POSTERIOR CALCULATION IN NORMAL MODEL

The following provides mathematical details of the prior-to-posterior mapping in normal models. This is a slightly different way to deliver the basic updating result outlined in DeGroot, Theorem 3 on p324 etc. Be on top on the ideas underlying the theory, and fill in the missing details.

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### Updating based on one observation

First, updating the prior $\theta \sim N(\mu_0, \tau_0^2)$ based on a single observation $(x|\theta) \sim N(\theta, \sigma^2)$. The posterior density is proportional to

$$p(\theta|x) \propto \exp(-Q(\theta)/2)$$

where

$$Q(\theta) = (x-\theta)^2/\sigma^2 + (\theta-\mu_0)^2/\tau_0^2 = \theta^2(1/\sigma^2 + 1/\tau_0^2) - 2\theta(x/\sigma^2 + \mu_0/\tau_0^2) + c$$

where $c$ does not depend on $\theta$. This equals

$$\theta^2/\tau_1^2 - 2\theta\mu_1/\tau_1^2 + c$$

with $\tau_1^2 = 1/(1/\sigma^2 + 1/\tau_0^2)$ and $\mu_1 = \tau_1^2(x/\sigma^2 + \mu_0/\tau_0^2)$. This further simplifies by "completing the square" as

$$(\theta-\mu_1)^2/\tau_1^2 - \mu_1^2/\tau_1^2 + c$$

or

$$Q(\theta) = (\theta-\mu_1)^2/\tau_1^2 + k$$

where $k$ does not depend on $\theta$. Hence

$$p(\theta|x) \propto \exp(-Q(\theta)/2) \propto \exp(-[(\theta^2-\mu_1)^2/\tau_1^2 + k]/2) \propto \exp(-(\theta-\mu_1)^2/(2\tau_1^2))$$

and we recognise this as the required normal density function, apart from the normalisation constant which we can infer is just $1/\sqrt{2\pi\tau_1^2}$.

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### Updating based on $n$ observations

Here we have a likelihood function

$$\prod_{i=1}^{n} p(x_i|\theta) \propto \exp(-Q_n(\theta)/2)$$

where

$$Q_n(\theta) = \sum_{i=1}^{n}(x_i-\theta)^2/\sigma^2.$$

Expand the sum of squares here as

$$\sum_{i=1}^{n}(x_i^2 - 2x_i\theta + \theta^2) = \sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}2x_i\theta + \sum_{i=1}^{n}\theta^2$$

which reduces to

$$n\theta^2 - 2n\theta\bar{x} + c = n(\theta^2 - 2\theta\bar{x}) + c$$

for some constant $c$ not depending on $\theta$. Completing the square here reduces this to a constant plus $n(\theta-\bar{x})^2$ so that $Q_n(\theta) = n(\theta-\bar{x})^2/\sigma^2 + k$ and the likelihood function is proportional to

$$\exp(-Q_n(\theta)/2) \propto \exp(-(\theta-\bar{x})^2/(2\sigma^2/n)).$$

This has the form of the likelihood function in the single observation case above – just replace the single observation $x$ by $\bar{x}$ and $\sigma^2$ by $\sigma^2/n$. Then you can simply quote the result directly with these changes, and the required formulæ for $\mu_n$ and $\tau_n^2$ drop out.

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