Midterm Examination # 2

Mth 135 = Sta 104

Tuesday, November 18, 2008
10:05 – 11:20 am
Version *b*

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes.
- **Show** your **work**. Neatness counts.
- Numerical answers: **four significant digits** or fractions **in lowest terms**.
- Extra worksheet and pdf & normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

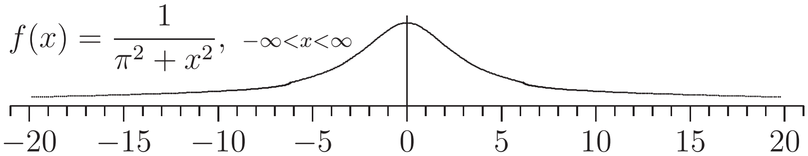
Signature: ____________________

Print **Name**: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: A random variable X has probability density function given by

$$f(x) = \frac{1}{\pi^2 + x^2}, \quad -\infty < x < \infty$$



a) (5) Find the probability density function for $Y = X/\pi$:

$f_Y(y) =$

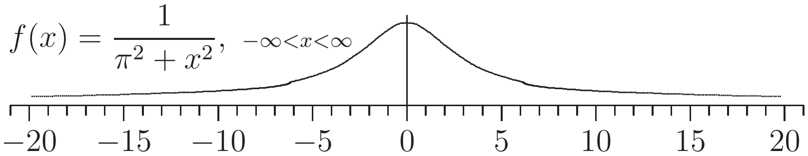
b) (5) Find the probability density function for $Z = \pi/X$:

$f_Z(z) =$

Problem 1 (cont):

The random variable X still has probability density function

$$f(x) = \frac{1}{\pi^2 + x^2}, \quad -\infty < x < \infty$$



c) (6) For each fixed $t > 0$ set $T = \begin{cases} |X| & \text{if } -t \le X \le t \\ 0 & \text{else} \end{cases}$,

the absolute value of X when $|X| \le t$ and otherwise zero. Find the expected value of this positive function of X:

$\mathsf{E}[\ T\ ] =$ ____________________

d) (4) Does X have a mean? Circle one: Y N

If so, what is it? If not, why not? $\mathsf{E}[X] =$ ____________________

Problem 2: A normally-distributed random variable $X \sim \mathsf{No}(\mu, \sigma^2)$ satisfies

$$\mathsf{P}[X \le 20] = \frac{1}{10} = \mathsf{P}[X > 100]$$

a) (5) What are the mean and variance? (Hint: You can get the mean by symmetry, then the standard deviation is easy)

$\mu_X =$ ________________ $\sigma_X^2 =$ ________________

b) (5) Find the indicated probabilities. Show your work:

$\mathsf{P}[X \le 60] =$ ________________ $\mathsf{P}[X^2 > 2500] =$ ________________

Problem 2 (cont):

c) (5) Give the mean and variance of $Y = 100 - 2X$:

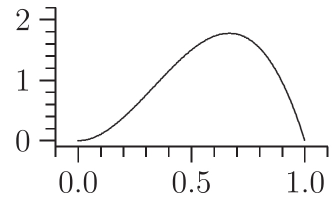
$\mu_Y =$ ________________ $\sigma_Y^2 =$ ________________

d) (5) Give the expectation and the name of the probability distribution for $Z = (X - \mu_X)^2/\sigma_X^2$:

$\mathsf{E}[Z] =$ ________________ Name: ________________

Problem 3: The random variable X has a Beta distribution with pdf

$$f(x) = \begin{cases} 12\,x^2(1-x) & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$



a) (4) Find the *mode* of this distribution, *i.e.*, the location m where $f(x)$ attains its maximum value, $f(m)$. Show your work.

$m =$ ____________________

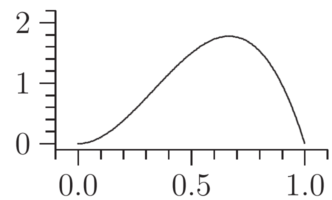
b) (8) Find the mean, variance of the indicator rand. vbl. $Y = \begin{cases} 1 & \text{if } X \le \frac{1}{2} \\ 0 & \text{if } X > \frac{1}{2} \end{cases}$:

$\mu = \mathsf{E}[Y] =$ ______________ $\sigma^2 = \mathsf{Var}[Y] =$ ______________

Problem 3 (cont):

The random variable X still has a Beta distribution with pdf

$$f(x) = \begin{cases} 12\,x^2(1-x) & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$

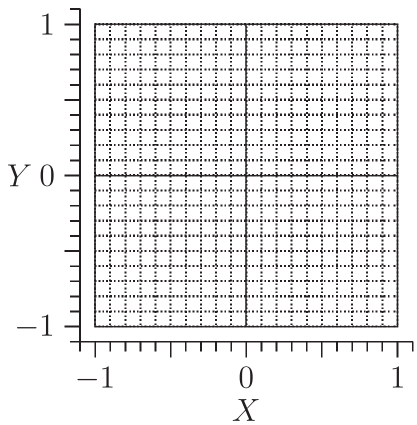


c) (8) Find the mean and variance of the random variable $Z = 1/X$:

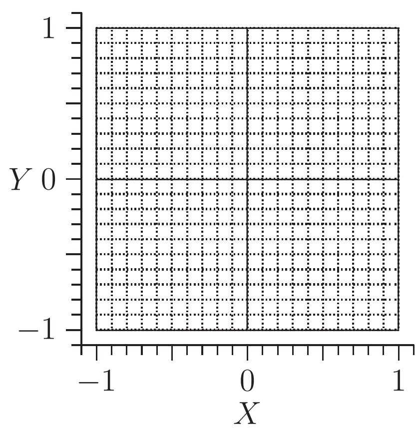
$\mu = \mathsf{E}[Z] =$ ____________ $\sigma^2 = \mathsf{Var}[Z] =$ ____________

x) (+2) Find the expectation of X^p for $X \sim \mathsf{Be}(\alpha, \beta)$, for all $p \in \mathbb{R}$ and $\alpha, \beta > 0$.

Problem 4: The random variables X and Y are independent, each with the uniform distribution on $[-1, 1]$. Draw **and shade** the indicated region on the plot and evaluate the probabilities requested (**no integration** is needed):

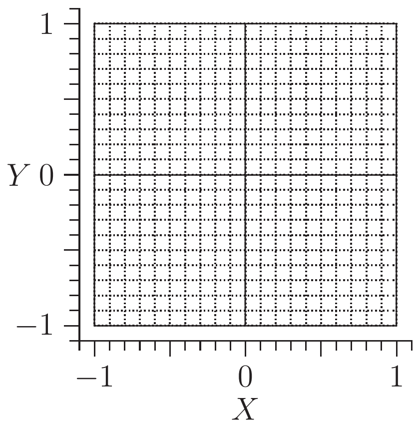
a) (4) 

$\mathsf{P}[X^2 < Y^2] =$ ______________

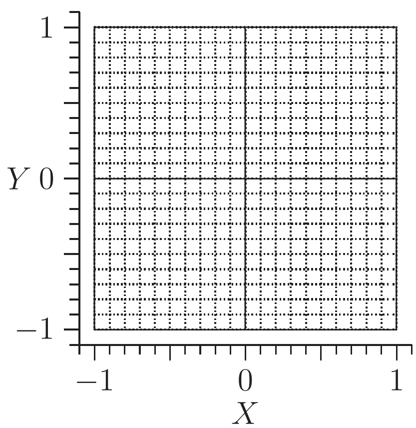
b) (4) 

$\mathsf{P}[X^2 + Y^2 < 0.64] =$ ______________

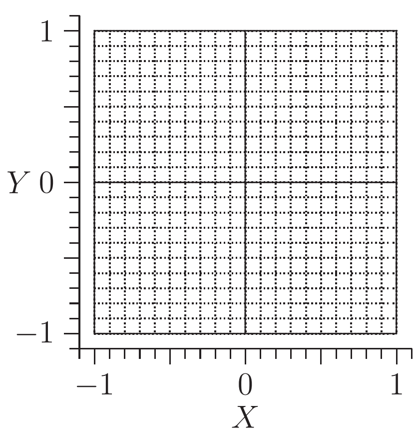
Problem 4 (cont):

c) (4) 

$\mathsf{P}\big[\ \sin(\pi X)<Y\big] =$ ______________

c) (4) 

$\mathsf{P}\big[\ |X-Y| < 1\big] =$ ______________

d) (4) 

$\mathsf{P}[\max(X, Y) > 0.5] =$ ______________

Problem 5: The random variables X and Y have joint probability density function

$$f(x,y) = \begin{cases} 8\,x\,y & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) (4) Find the marginal p.d.f. for X, correctly for *all* values of x:

$$f_X(x) = \begin{cases} \\ \\ \end{cases}$$

b) (4) Find the marginal density function for Y:

$$f_Y(y) = \begin{cases} \\ \\ \end{cases}$$

Problem 5 (cont):

X and Y still have joint pdf $f(x, y) = 8\,x\,y$ on $0 < x < y < 1$:

c) (4) Find the density function for $Z = \sqrt{Y}$:

$$f_Z(z) = \begin{cases} \\ \\ \end{cases}$$

d) (4) Find the density function for $R = X/Y$:

$$f_R(r) = \begin{cases} \\ \\ \end{cases}$$

e) (4) All but one of the distributions in parts a)–d) is from the same family of distributions. Without doing any unnecessary integration, give all four means:

$\mathsf{E}[X] =$ ______ $\mathsf{E}[Y] =$ ______ $\mathsf{E}[Z] =$ ______ $\mathsf{E}[R] =$ ______

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

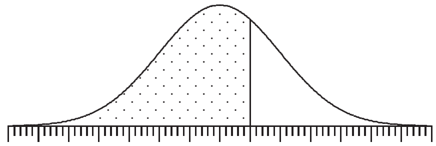


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,G,B)$	$f(x)=\frac{\binom{G}{x}\binom{B}{n-x}}{\binom{G+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{G}{G+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	

Name: ______________________

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