Midterm Examination # 1

Mth 135 = Sta 104

Thursday, October 19, 2008
10:05 – 11:20 am
Version *a*

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes.
- **Show** your **work**. Neatness counts.
- Numerical answers: **four significant digits** or fractions **in lowest terms**.
- PDF and normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ______________________

Print Name: ______________________

1.	/20
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6.	/20
Total:	/120

Problem 1: Lisa tosses three fair coins into the air.

a) (10) Find the probability mass function (pmf) for X, the (total) number of heads *minus* the number of tails:

$p(x) = \mathsf{P}[X = x] =$

b) (5) Find the expectation of X:

$\mathsf{E}[X] =$

c) (5) Find the expectation of $Y := 2^X$:

$\mathsf{E}[Y] =$

Problem 2: Repeatedly, three identical fair coins are thrown simultaneously; each trial we note whether or not they all show the same face.

a) (10) What is the probability that the number X of *trials* needed before they all show the same face is (strictly) **more than** two? Give exact numerical answer.

$\mathsf{P}[X > 2] =$ ______________________

b) (10) What is the probability that the number Y of times they all agree in the next 300 trials is at least 68? Half credit for a correct sum or integral expression; full credit for normal-distribution approximation to four correct decimal places.

(10) $\mathsf{P}[Y \geq 68] \approx$ ______________________

Problem 3: Two events, A and B, have probabilities $\mathsf{P}[A] = 0.5$ and $\mathsf{P}[B] = 0.3$ respectively. Find the probabilities asked, under the conditions given; each "If" condition applies *only* to that question.

a) (5) If A and B are independent, then $\mathsf{P}[A \cap B] =$ ______________.

b) (5) If A and B are exclusive, then $\mathsf{P}[A \cup B] =$ ______________.

c) (5) If $\mathsf{P}[A \cup B] = 0.74$, then $\mathsf{P}[A \mid B] =$ ______________.

d) (5) If $\mathsf{P}[A \mid B] = 0.40$, then $\mathsf{P}[A \cup B] =$ ______________.

Problem 4: Two points each, no explanations needed:

a) If X, Y indep., then $\mathsf{Var}[X - Y] = \mathsf{Var}[X] - \mathsf{Var}[Y]$ T F

b) If $\mathsf{E}[X] = 4$ and $X \geq 2$, then $\mathsf{P}[X \geq 10] \leq 0.25$. T F

c) If X and Y are independent, then $\mathsf{E}[X/Y] = \mathsf{E}[X]/\mathsf{E}[Y]$ T F

d) If X is Poisson & if $\mathsf{P}[X = 1] = \mathsf{P}[X = 2]$, find $\mathsf{E}[X] =$ __________

e) If $\mathsf{E}[X] = 3$, $\mathsf{Var}[X] = 4$, $Y = 2X + 1$, then $\mathsf{E}[Y^2] =$ __________

f) If $a \in \mathbb{R}$ and $r \neq 1$, find & simplify: $\sum_{k=2}^{9} a\, r^k =$ __________

g) If $\mathsf{P}[X = k] = \frac{k}{10}$ for $k \in \{0, 1, 2, 3, 4\}$ find: $\mathsf{Var}[X] =$ __________

h) If $\mathsf{E}[X] > \mathsf{E}[Y]$ then $\mathsf{P}[X > 2] > \mathsf{P}[Y > 2]$ T F

i) If $\mathsf{P}[X > Y] = 1$ then $\mathsf{E}[X] > \mathsf{E}[Y]$ T F

j) If $\mathsf{P}[A] = \mathsf{P}[B]$ then $\mathsf{P}[A \mid B] = \mathsf{P}[B \mid A]$ T F

Problem 5: Every day when Homer spills his beer it makes a circular stain on the floor. The areas X_i (in cm^2) of these circular stains are random variables all with the Exponential distribution with rate $\lambda = 0.10$:

$$\mathsf{P}[X > x] = e^{-x/10} \qquad \mathsf{E}[X_i] = \mu = 10\,\text{cm}^2 \qquad \mathsf{Var}[X_i] = \sigma^2 = 100\,\text{cm}^4.$$

Consider the sum of these areas: $S = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$. Give numerical answers below to four decimals **with units**; show your work.

a) (5) Without any more information, is it possible to find the mean $\mathsf{E}[S]$? If so, find it; if not, explain why: $\mathsf{E}[S] =$ ____________________

b) (5) Without any more information, is it possible to find the variance $\mathsf{Var}[S]$? If so, find it; if not, explain why: $\mathsf{Var}[S] =$ ____________________

Problem 5 (cont):

As before, $S = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$ with $\{X_j\} \sim \mathsf{Ex}(0.10)$.

c) (5) Find the probability that the *diameter* D_1 of Monday (Day #1)'s stain exceeds 4 cm:

$\mathsf{P}[D_1 > 4] =$ ______________

d) (5) If $\{X_j\}$ are independent, give the name, mean, and variance of the distribution for S:

Name of distribution: ______________

$\mathsf{E}[S] =$ ______________ $\mathsf{Var}[S] =$ ______________

Problem 6: A hat contains $n = 5$ coins, four of which are fair (so that $\mathsf{P}[H] = 1/2$) and one of which is biased with $\mathsf{P}[H] = 4/5$. A single coin is drawn at random from the hat. The questions below are about this one coin; it is not replaced, and no other coin is drawn. Be sure to **simplify** your answers below. F and B denote the events that the coin is Fair and Biased, respectively, and H_i the event that it falls Heads on the i^{th} toss.

a) (10) On the first toss, it lands Heads. What is the (conditional) probability that it is a fair coin?

   $\mathsf{P}[F \mid H_1] =$ ____________________

b) (10) It is tossed a second time. What is the (conditional) probability that it will land Heads this time, too, given Heads on the first toss? (note it could be fair or biased)

   $\mathsf{P}[H_2 \mid H_1] =$ ____________________

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz:$$

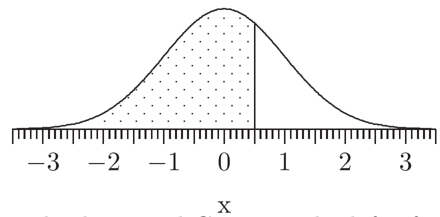

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\, e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,G,B)$	$f(x)=\frac{\binom{G}{x}\binom{B}{n-x}}{\binom{G+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{G}{G+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	