Final Examination

Mth 135 = Sta 104

Sunday, 2008 December 14, 2:00 – 5:00pm

- This is a **closed book** exam— please put your books on the floor.
- You may use a calculator and **two pages** of your own notes.
  Do not share calculators or notes.
- Please ask me questions if a problem needs clarification.
- **Show** your **work**. Boxing answers helps me find them.
- Numerical answers: **four significant digits** or fractions **in lowest terms**. Simplify expressions.
- Normal distribution and pdf/pmf tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ____________________

Print Name: ____________________

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
Total:			/200

Problem 1: William Shatner and Chuck Norris love to compete with each other, in everything from starring in cheesy TV action series to selling stuff on TV. Let X be the number of rerun episodes of Shatner's series *Star Trek* shown on cable TV in a week, and let Y be the number of episodes of Norris's *Walker, Texas Ranger*. Suppose that X and Y have independent Geometric distributions with possible values $0, 1, 2, ...$ and means $\mathsf{E}X = 9$ and $\mathsf{E}Y = 4$.

a) (5) Find the probability that no epsiode of either show appears in a week:

$\mathsf{P}[X = 0,\ Y = 0] =$ ____________________

b) (5) Find the probability that both shows air the same number of times in a week:

$\mathsf{P}[X = Y] =$ ____________________

Problem 1 (cont):

c) (5) Let $Z = \min(X, Y)$. What is the probability that both shows are shown at least 7 times?

$\mathsf{P}[Z \geq 7] =$ ____________________

d) (5) Find the expected value of $Z = \min(X, Y)$, without doing any sums or integrals:

$\mathsf{E}[Z] =$ ____________________

Problem 2: For 2pt each, write your answers in the boxes provided. Each c is some unspecified constant. No problem requires (much) integration. Full solutions aren't required, but show a sketch of the idea.

a) X has pdf $f(x) = c\,x^3\,e^{-8x}$, $x > 0$. What is $\mathsf{E}X$?

b) If $X \sim \mathsf{Un}(2, 10)$, what is the dist'n of $Y \equiv 3 - X/2$?

c) If $X \sim \mathsf{No}(2, 4)$, what is the dist'n of $Y \equiv 3 - X/2$?

d) If $X \sim \mathsf{Ex}(4)$, what is the dist'n of $Y \equiv X/2$?

e) If $X \sim \mathsf{Ex}(4)$, what is the median of X?

f) If $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp C$, is $A \perp\!\!\!\perp C$?

g) $X \sim \mathsf{Un}(-2, 4)$. Find $\mathsf{P}[X^2 > 1]$.

h) If $X, Y \sim \mathsf{Po}(3.5)$ and $X \perp\!\!\!\perp Y$, what dist'n does $X + Y$ have?

i) If $X, Y \sim \mathsf{No}(1, 2)$ are indep, what is $\mathsf{Cov}[(X + Y), (X - Y)]$?

j) $\mathsf{Cov}[X, Y] = 3$ and $\mathsf{E}[X] = 1$. What is $\mathsf{Cov}[X, 2 + Y]$?

Problem 3: The number of times William Shatner's silly commercial for Priceline.com appears on cable TV each hour is a random variable X with a Poisson distribution with mean $\mathsf{E}X = 1/2$. For 4 pts each, find the following:

a) $\mathsf{E}[2X] =$ ____________________

b) $\mathsf{E}[X^2] =$ ____________________

c) $\mathsf{E}[2^X] =$ ____________________

d) $\mathsf{E}[X!] =$ ____________________

e) $\mathsf{Var}[(10 - 4X)] =$ ____________________

Problem 4: The random variables X and Y have joint density function $f(x, y)$ on the plane $\mathbb{R}^2 = \{(x, y) : \ -\infty < x < \infty, \ -\infty < y < \infty\}$. Give the simplest expressions you can for each of the following calculations. Most will require one or two integral expressions; some have more than one correct answer (any one will do). Be clear and careful about limits of integration. William and Chuck don't know much calculus...

a) $\mathsf{P}[1 < X < 10] =$

b) $\mathsf{P}[X^2 < Y] =$

c) $\mathsf{E}[(X - Y)^2] =$

d) $\mathsf{P}[X > 0 \mid Y > 0] =$

e) $\mathsf{E}[X \mid Y > 0] =$

Problem 5: Chuck Norris starred in such forgettable movies as *The Cutter* (2005) and *Bells of Innocence* (2003). The times between successive Chuck Norris movies (in years) are independent random variables T_1, T_2,... with exponential distributions with rate $\lambda = 1/2$, so:

$$\mathsf{P}[T_n > t] = e^{-t/2}, \qquad t > 0$$

then:

a) (5) What are the distributions of the times *in months* between Norris films? Give the name of the distribution and the value(s) of any parameter(s):

b) (5) The time until Chuck completes five more films is the sum

$$S_5 = T_1 + T_2 + T_3 + T_4 + T_5$$

What is the expected *square* of this sum? (specify the units!)

$\mathsf{E}\left[S_5^2\right] =$ ____________________

c) (5) What is the probability distribution of S_5? Give its name and the value(s) of any parameter(s):

d) (5) What is the (exact) probability distribution for the number of films Chuck makes in a decade? Give its name and the value(s) of any parameter(s):

Problem 6: Chuck and William are puzzling about three events A, B, and C. Each event has the same probability, p, which might take *any* value between zero and one (for example, it might be more than 1/3).

a) (4) If they are independent, find the probability of the intersection as a function of p, valid for all $0 < p < 1$:

$$\mathsf{P}[A \cap B \cap C] \; = \; \underline{\hspace{4cm}}$$

b) (4) If they are independent, find the probability of the union, as a function of p:

$$\mathsf{P}[A \cup B \cup C] \; = \; \underline{\hspace{4cm}}$$

c) (6) Find the smallest and largest possible values for the probability of the union, as functions of p, if they are not known to be independent:

$$\underline{\hspace{4cm}} \; \le \; \mathsf{P}[A{\cup}B{\cup}C] \; \le \; \underline{\hspace{4cm}}$$

d) (6) Find the smallest and largest possible values for the probability of the intersection, as functions of p, without assuming independence:

$$\underline{\hspace{4cm}} \; \le \; \mathsf{P}[A{\cap}B{\cap}C] \; \le \; \underline{\hspace{4cm}}$$

Problem 7: Let $X, Y, Z \sim \mathsf{No}(0,1)$ be independent normally distributed random variables, each with mean 0 and variance 1. Calculate to four decimal places:

a) (5) $\mathsf{P}[|2X| < 1,\ |Y| < 1,\ |\frac{1}{2}Z| < 1] =$ ____________
(the probability that all three inequalities hold, simultaneously)

b) (5) $\mathsf{E}[(X + 2Y + 3Z)^2] =$ ____________

c) (5) $\mathsf{P}[X + 2Y < 3Z + 3.75] =$ ____________

d) (5) $\mathsf{P}[5X + 3Y < 1 \mid 3X + 4Z = 5] =$ ____________
Note: This is a question about the conditional *distribution* of the random variable $W_2 = 5X + 3Y$, given the value of another random variable $W_1 = 3X + 4Z$. Think about the joint dist'n of $W_1,\ W_2$.

Problem 8: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, or *Uniform* and, whatever the distribution, give its mean μ:

a) The number of times Chuck Norris rolls a fair six-sided die before (and not including) the first roll of a six:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

b) In a random grab of 3 from a boxed set of all ten of William's Star Trek movies, the number of films featuring Romulans if if five of the ten films feature Romulans, 8 of the ten feature Klingons, and two of the ten feature travel through time:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

c) The number of jurors on the 120-person Academy Awards jury who vote in favor of an Oscar for Chuck Norris, if 5% of all AMPAS members support him and if the jury is selected randomly from the thousands of AMPAS members:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

d) The number of sarcastic remarks Chuck Norris makes in an episode of *Walker, Texas Ranger* if he makes one about every five minutes, if remarks in disjoint time intervals are disjoint, and if an episode lasts 25 minutes (plus commercials)?
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

e) The total number of silly grins William Shatner makes in his role as veteran police sergeant T. J. Hooker in one 26-episode season, if he makes a geometrically-distributed number of silly grins a show with an average of ten per show?
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

Problem 9: Chuck and William are puzzling again about three events events A, B, and C; this time they are all independent, but they have different probabilites:

$$\mathsf{P}[A] = 1/4 \qquad \mathsf{P}[B] = 1/2 \qquad \mathsf{P}[C] = 3/4$$

Let N be the number of events that occur— a random variable taking the values 0, 1, 2, or 3.

a) (5) Does N have the Binomial distribution? If so, what are the parameters? If not, why not?

b) (5) Find the probability:

$\mathsf{P}[N = 1] =$ ____________________

c) (5) Find the conditional probability:

$\mathsf{P}[A \mid N = 1] =$ ____________________

d) (5) Find the expectations:

$\mathsf{E}[N] =$ ____________________ $\mathsf{E}[2^N] =$ ____________________

Problem 10: Let X be an exponentially distributed random variable with parameter $\lambda = 4$ (so $\mathsf{E}[X] = 1/4$) and define the random variable Y by $Y = 1/X$.

a) $f_Y(y) =$ ______________________
Determine the probability density function (pdf) for Y, at all real numbers y.

b) $\mathsf{P}[0.25 \le Y \le 2] =$ ____________
Compute the indicated probability as a decimal, correct to four places. (**Hint**: this is easy and does not depend on a) above).

c) $\mathsf{P}[X < Y] =$ ____________
Compute the indicated probability as a decimal, correct to four places.

d) $\mathsf{E}[e^X] =$ ____________
Find the expected value of $\exp(X) = \exp(1/Y)$ (**not** of e^Y!)

Done!

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

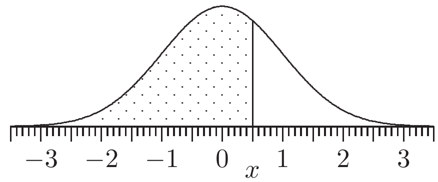

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,G,B)$	$f(x)=\frac{\binom{G}{x}\binom{B}{n-x}}{\binom{G+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{G}{G+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\mu)$	$f(x)=\frac{\mu^x}{x!}e^{-\mu}$	$x\in\mathbb{Z}_+$	μ	μ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	