# Midterm Examination # 2

Mth 135 = Sta 104

Thursday, November 17, 2005
10:05 – 11:20 am
Version $b$

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes.
- **Show** your **work**. Neatness counts.
- Numerical answers: **four significant digits** or fractions **in lowest terms**.
- PDF and normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors, nor will I accept the actions of those who do.
- I will conduct myself responsibly and honorably in all my activities as a Duke student.

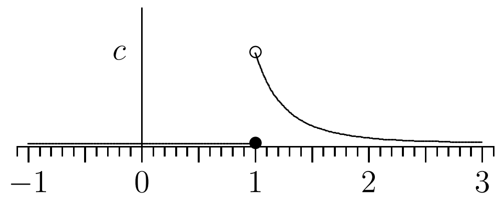
Signature: ____________________

Print Name: ____________________

| | |
|---|---|
| 1. | /20 |
| 2. | /20 |
| 3. | /20 |
| 4. | /20 |
| 5. | /20 |
| Total: | /100 |

1. A random variable $X$ has probability density function given by

$$f(x) = \begin{cases} c/x^4 & x > 1 \\ 0 & x \le 1 \end{cases}$$




a) (5) What is the value of $c > 0$?

$c =$ ____________________

b) (5) Find the CDF, correct at every point:

$$F(x) = \begin{cases} \\ \\ \end{cases}$$

c) (5) Find the expectation of the power $X^p$, for every $p \in \mathbb{R}$:

$$\mathsf{E}[X^p] = \begin{cases} \\ \\ \end{cases}$$

d) (5) What is the variance of $X$?

$\mathsf{Var}[X] =$ ____________________

2. A normally-distributed random variable $X \sim \mathsf{No}(\mu, \sigma^2)$ satisfies

$$\mathsf{P}[X \le -4] = \frac{1}{3} = \mathsf{P}[X > 6]$$

a) (5) What are the mean and variance?

$\mu_X =$ ________________ $\sigma_X^2 =$ ________________

b) (5) Find the indicated probabilities:

$\mathsf{P}[X \le 2] =$ ________________ $\mathsf{P}[X^2 > 1] =$ ________________

c) (5) Give the mean and variance of $Y = 2X - 12$:

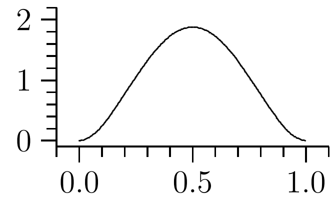
$\mu_Y =$ ________________ $\sigma_Y^2 =$ ________________

d) (5) Find the indicated probabilities:

$\mathsf{P}[Y \le 2] =$ ________________ $\mathsf{P}[2/Y > 1] =$ ________________

3. The random variable $X$ has a Beta distribution with pdf

$$f(x) = \begin{cases} 30\,x^2(1-x)^2 & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$




a) (5) Without doing any integration, give the mean and variance:

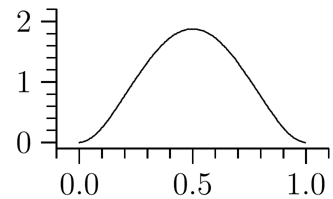
$\mu_X =$ ________________ $\sigma^2_X =$ ________________

b) (5) Find the pdf for $Y = \frac{X}{1-X}$:

$$f_Y(y) = \begin{cases} \\ \\ \end{cases}$$

3. (Cont'd) The random variable $X$ still has a Beta distribution with pdf

$$f(x) = \begin{cases} 30\,x^2(1-x)^2 & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$




c) (5) Find the pdf for $Z = -\ln X$:

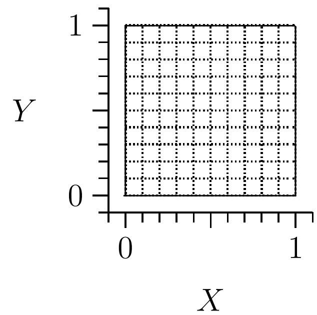
$$f_Z(z) = \begin{cases} \\ \\ \end{cases}$$

d) (5) Are $Y$ and $Z$ independent? Prove your answer.[1]

[1]Warning: Note that $S \equiv (2X-1)/|2X-1|$ and $Z \equiv X(1-X)$ *are* independent, so "$Y$ and $Z$ are both functions of $X$" is *not* a correct answer!

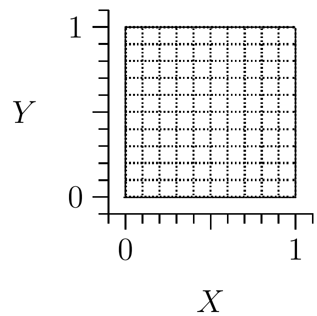
4. The random variables $X$ and $Y$ are independent, each with the uniform $\mathsf{Un}(0,1)$ distribution. Draw **and shade** the indicated region on the plot and evaluate the probabilities requested:

a) (5)



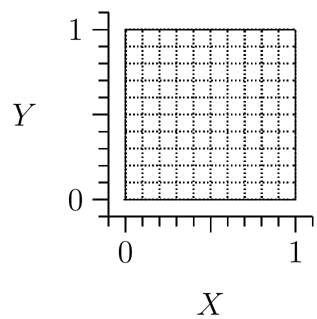

$\mathsf{P}[\frac{X}{Y}<2] =$ ______________

b) (5)



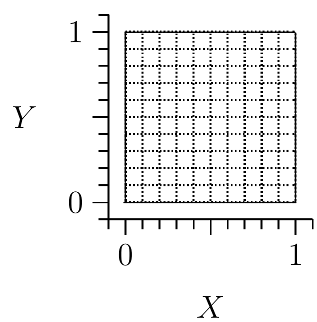

$\mathsf{P}[X^2 + Y^2 < 0.64] =$ ______________

c) (5)




$\mathsf{P}[X > Y^2] =$ ______________

d) (5)




$\mathsf{P}[\max(X,Y) > 0.8] =$ ______________

5. The random variables $X$ and $Y$ have joint probability density function

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

a) (5) Find the marginal density functions:

$$f_X(x) = \begin{cases} \\ \\ \end{cases} \qquad\qquad f_Y(y) = \begin{cases} \\ \\ \end{cases}$$

b) (5) Are $X$ and $Y$ independent? Prove your answer.

5. (Cont'd) The random variables $X$ and $Y$ still have joint pdf

$$f(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

c) (5) Find the density function for $R = X/Y$:

$$f(r) = \begin{cases} \\ \\ \end{cases}$$

d) (5) Are $Y$ and $R$ independent? Prove your answer.

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

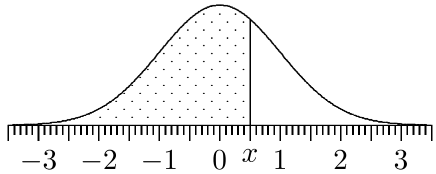



**Table 5.1** Area $\Phi(x)$ under the Standard Normal Curve to the left of $x$.

| $x$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

| Name | Notation | pdf/pmf | Range | Mean $\mu$ | Variance $\sigma^2$ | |
|---|---|---|---|---|---|---|
| **Beta** | $\mathsf{Be}(\alpha,\beta)$ | $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | $x\in(0,1)$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | |
| **Binomial** | $\mathsf{Bi}(n,p)$ | $f(x)=\binom{n}{x}p^x q^{(n-x)}$ | $x\in\{0,\cdots,n\}$ | $n\,p$ | $n\,p\,q$ | $(q=1-p)$ |
| **Exponential** | $\mathsf{Ex}(\lambda)$ | $f(x)=\lambda\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $1/\lambda$ | $1/\lambda^2$ | |
| **Gamma** | $\mathsf{Ga}(\alpha,\lambda)$ | $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$ | $x\in\mathbb{R}_+$ | $\alpha/\lambda$ | $\alpha/\lambda^2$ | |
| **Geometric** | $\mathsf{Ge}(p)$ | $f(x)=p\,q^x$ | $x\in\mathbb{Z}_+$ | $q/p$ | $q/p^2$ | $(q=1-p)$ |
| | | $f(y)=p\,q^{y-1}$ | $y\in\{1,\cdots\}$ | $1/p$ | $q/p^2$ | $(y=x+1)$ |
| **HyperGeo.** | $\mathsf{HG}(n,A,B)$ | $f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$ | $x\in\{0,\cdots,n\}$ | $n\,P$ | $n\,P\,(1-P)\frac{N-n}{N-1}$ | $(P=\frac{A}{A+B})$ |
| **Logistic** | $\mathsf{Lo}(\mu,\beta)$ | $f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\pi^2\beta^2/3$ | |
| **Log Normal** | $\mathsf{LN}(\mu,\sigma^2)$ | $f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}_+$ | $e^{\mu+\sigma^2/2}$ | $e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$ | |
| **Neg. Binom.** | $\mathsf{NB}(\alpha,p)$ | $f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$ | $x\in\mathbb{Z}_+$ | $\alpha q/p$ | $\alpha q/p^2$ | $(q=1-p)$ |
| | | $f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$ | $y\in\{\alpha,\cdots\}$ | $\alpha/p$ | $\alpha q/p^2$ | $(y=x+\alpha)$ |
| **Normal** | $\mathsf{No}(\mu,\sigma^2)$ | $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ | $x\in\mathbb{R}$ | $\mu$ | $\sigma^2$ | |
| **Pareto** | $\mathsf{Pa}(\alpha,\beta)$ | $f(x)=\beta\,\alpha^\beta/x^{\beta+1}$ | $x\in(\alpha,\infty)$ | $\frac{\alpha\,\beta}{\beta-1}$ | $\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$ | |
| **Poisson** | $\mathsf{Po}(\lambda)$ | $f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$ | $x\in\mathbb{Z}_+$ | $\lambda$ | $\lambda$ | |
| **Student** $t$ | $t(\nu)$ | $f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$ | $x\in\mathbb{R}$ | $0$ | $\nu/(\nu-2)$ | |
| **Unif. (Cont)** | $\mathsf{Un}(a,b)$ | $f(x)=\frac{1}{b-a}$ | $x\in(a,b)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | |
| **Unif. (Disc)** | $\mathsf{Un}\{a,\cdots,b\}$ | $f(x)=\frac{1}{b-a+1}$ | $x\in\{a,\cdots,b\}$ | $\frac{a+b}{2}$ | $\frac{(b-a)(b-a+2)}{12}$ | |
| **Weibull** | $\mathsf{We}(\alpha,\beta,\gamma)$ | $f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$ | $x\in(\gamma,\infty)$ | $\gamma+\beta\Gamma(1+\alpha^{-1})$ | | |

Name: ____________

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