Midterm Examination # 1

Mth 135 = Sta 104

Thursday, October 7, 2004
10:05 – 11:20 am

This is a closed-book exam so please do not refer to your notes or text. You may use a single one-sided sheet of *your own* notes, if you wish, but you may not share materials. A normal distribution table, the PDF handout, and a blank worksheet are attached to the exam. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other. You may use a calculator but not a laptop computer or PDA.

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. You should spend about 10 minutes on each problem. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors, nor will I accept the actions of those who do.
- I will conduct myself responsibly and honorably in all my activities as a Duke student.

Signature: ____________________

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

1. In the hope of increasing the chance of rolling a "seven," an unscrupulous gambler has added weights to the six on one die and to the ace on the other, leading to the following probabilities:

Face:	1	2	3	4	5	6
X_1 (red die):	0.5	0.1	0.1	0.1	0.1	0.1
X_2 (blue die):	0.1	0.1	0.1	0.1	0.1	0.5

The faces shown by the two dice are independent. If the gambler rolls this pair of dice,

a. (6) Find the probability that the sum of the two faces shown is seven:

$\Pr[X_1 + X_2 = 7] =$

b. (6) Find the probability that the two dice show the same face:

$\Pr[X_1 = X_2] =$

c. (8) One of the dice is chosen at random, and rolled, yielding a six. What is the probablity that it is the Red die?

$\Pr[I = 1 | X_I = 6] =$

2. In a certain (imaginary) lake, the number X_t of fish caught in t hours has the Poisson distribution with mean $\mu = 4t$,

$$\Pr[X_t = k] = e^{-4t}\frac{(4t)^k}{k!}$$

for $k = 0, 1, \ldots.$

a. (10) What is the probability of catching at least two fish in the first fifteen minutes? Give your answer as a decimal approximation, accurate to at least two decimal places, or as a *finite* sum.

   $\Pr[X_{.25} \geq 2] =$

b. (10) What is the probability that the length of time T_1 until catching the first fish is more than one-half hour? (HINT: Try to express this *event* in terms of $X_{.5}$)

   $\Pr[T_1 > 0.5] =$

Note: $e^{-.25} = 0.7788$, $e^{-.5} = 0.6065$, $e^{-1} = 0.3679$, and $e^{-2} = 0.1353$.

3. A fair die is tossed repeatedly until an ace (i.e. one) appears; the random variable X is the **number of non-aces** that occur before that first ace. X has mean $\mu = q/p = 5$ and variance $\sigma^2 = q/p^2 = 30$. Find:

a. (4) The probability that X is odd:

$\Pr[X \in \{1, 3, 5, ...\}] =$

b. (12) The following expectations and variance:

$\mathsf{E}[(-1)^X] =$ ______________ $\mathsf{E}[4X + 7] =$ ______________

$\mathsf{E}[2^X] =$ ______________ $\mathsf{Var}[4X + 7] =$ ______________

c. (4) The CDF:

$F(x) = \Pr[X \le x] =$

4. A bowl contains six Red poker-chips and four Black ones. All ten chips are drawn from the bowl, one at a time, and their color is noted. Chips are not replaced once they have been drawn.

a. (5) What is the probability of the event that the first three chips are all the same color?

Pr[Match] =

b. (5) Given that the first three are the same color, what is the probability that the color is Red?

Pr[Red|Match] =

c. (10) What is the expected value of X, the number of Red chips drawn *before* any Black chips are drawn? (One way to do it: write $X = \sum I_i$, where the indicator I_i is **one** if the i^{th} Red chip is drawn before any of the four Black chips, and **zero** otherwise. What is $E[I_i]$?).

$\mathsf{E}[X] =$

5. Remove all twelve face-cards (J,Q,K) from a deck of cards, leaving forty cards $A, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Let X denote the number of aces in a draw of N cards from this deck.

a. (5) If we draw $N = 4$ cards *without* replacement, what is the exact probability of drawing at least one ace?

$P[X \geq 1] =$

b. (5) If we draw $N = 400$ cards *with* replacement, what is the **exact** probability of drawing at least 50 aces? (Give an expression, not a number)

$\Pr[X \geq 50] =$

c. (10) Give a **numerical approximation** to the probability in b. above, still with $N = 400$ and *with* replacement, accurate to at least three decimal places:

$\Pr[X \geq 50] \approx$

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

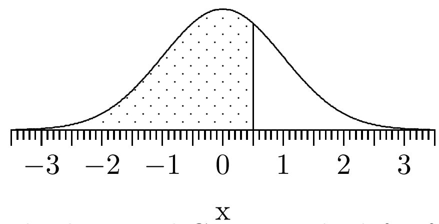

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$
$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	t_ν	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	