Final Examination (Ver A)

Mth 135 = Sta 104

Tuesday, 2004 December 7, 9:00am – 12:00n

This is a **closed-book** exam. You may use a calculator and two pages of *your own* notes, if you wish, but you may not share materials. If you don't understand something in one of the questions **please** ask me. The problems are not all equally difficult, but they all count equally.

You must **show** your **work** to get credit. Unsupported answers aren't acceptable, even if they're correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. It is to your advantage to write your solutions as clearly as possible.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. Please acknowledge the Duke Community Standard:

I have adhered to the Duke Community Standard in completing this examination.

Print: ____________________ Sign: ____________________

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
Total:			/200

Problem 1: True or false? Circle one (2 pts each, parts a–j).

a. T F For any two random variables X and Y,

$$\mathsf{E}[X-Y] = \mathsf{E}[X] - \mathsf{E}[Y].$$

b. T F For independent random variables X and Y,

$$\mathsf{Var}[X-Y] = \mathsf{Var}[X] - \mathsf{Var}[Y].$$

c. T F If X has density function $f_X(x)$ and if $Y = X^2$ then Y has density function $f_Y(y) = f_X(\sqrt{y})$ for all $y > 0$.

d. T F If X and Y are independent then $\mathsf{P}[X = Y] = 0$.

e. T F If X and Y are independent then $\mathsf{E}[X/Y] = \mathsf{E}[X]/\mathsf{E}[Y]$.

f. T F If X and Y have independent Uniform distributions on $[0, 1)$ then $X + Y$ has a Uniform distribution on $[0, 2)$

g. T F The function $f(a) = \mathsf{E}[(X-a)^2]$ takes its minimum value at $a = \mu \equiv \mathsf{E}[X]$, and the minimum is $f(\mu) = \sigma^2 = \mathsf{Var}[X]$.

h. T F If A and B are independent then $\mathsf{P}[A \cup B] = \mathsf{P}[A] + \mathsf{P}[B]$.

i. T F If A and B are independent then A^c and B^c are independent.

j. T F If $\mathsf{E}[X \cdot Y] = \mathsf{E}[X] \cdot \mathsf{E}[Y]$ then $\mathsf{Cov}[X, Y] = 0$, but X and Y might not be independent.

k. T F The answer to this question is ‘F’.

Problem 2: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, or *Uniform* and, whatever the distribution, give its mean μ:

a. (4) The number of arrows Robin Hood misses with before he finally hits the target, a small picture of the Sheriff of Nottingham, if shots are independent and hit with probability 0.01:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

b. (4) The total take from robbing 100 noblemen, if the heist takes are independent and average £10 per nobleman:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

c. (4) The number of aces Robin Hood throws in 24 throws of a fair 6-sided die:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

d. (4) The distance covered by the third-shortest flight of four arrows, if flights are independent and uniformly distributed from zero to one furlong:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

e. (4) The time until Maid Marian's third kiss, if kisses come about twice per hour and the numbers in disjoint time intervals are independent:
◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un
$\mu =$

Problem 3: Arrows arrive at Robin Hood's camp at an average rate of one every ten minutes. Find the probabilities of the following events, explaining briefly your assumptions:

a. (5) No arrow ariving in a given one-hour period:

b. (5) Between four and six arrows (inclusive) in a given one-hour period:

c. (10) Between 450 and 550 (inclusive) in a given 100-hour period (give answer as a decimal)

Problem 4: Each time Robin Hood fights, his sword gets a new scratch at a randomly-drawn point, X meters from the hilt, with probability density function given by

$$f(x) = \begin{cases} c\,x & 0 < x < 1 \\ 0 & x \notin (0,1) \end{cases}$$

for some number $0 < c < \infty$.

a. (5) What is the value of c? Why?

b. (5) Calculate the probability that $a \le X \le b$, for the values $a = -1.5$ and $b = 0.5$:

$\mathsf{P}[\,a \le X \le b\,] =$ __________________

c. (5) Find the *standard deviation* of X:

$\mathsf{SD} =$ __________________

d. (5) Find the p.d.f. for $Y = X^{-1}$:

$f_Y(y) = \Big\{$

Problem 5: Robin Hood is pretty good at shooting arrows, but Little John is better. In furlongs, Robin's arrows have a uniform distribution on the interval $[0, 1]$, while Little John's have an exponential distribution with mean one. If Robin and John each (independently) shoot arrows that go X and Y furlongs, respectively, then find:

a. (5) The probability that Robin's goes farther:

$\mathsf{P}[X > Y] =$ __________________

b. (5) The expected average flight distance:

$\mathsf{E}\left[\frac{X+Y}{2}\right] =$ ______________

c. (5) The indicated conditional probability:

$\mathsf{P}[X > Y | Y \le 1] =$ ______________

d. (5) The p.d.f. for $Z = \min(X, Y)$, the minimum of the two arrow flights:

$f(z) = \Bigg\{$

Problem 6: Three of Robin Hood's four vassals always tell the truth; the fourth one lies 50% of the time. Robin chooses a vassal at random and asks a question. All questions concern this one vassal; no other is chosen.

a. (5) What is the probability of getting a true answer?

b. (5) If the answer is true, what is the probability that the vassal is always-truthful?

c. (5) If the answer is true, what is the probability that this vassal will answer the next question truthfully as well?

d. (5) If Robin asks one vassal N questions, and gets a true answer for each, how large must N be for the conditional probability that the vassal is truthful to be at least 99%?

Problem 7: Each year Robin Hood poaches random weights X of deer and Y of rabbits from Sir Guy of Gisbourne's manor grounds. These have normal probability distributions that satisfy:

$$\mathsf{E}[X] = 100 \quad \mathsf{E}[Y] = 10 \quad \mathsf{E}[X^2] = 10,625 \quad \mathsf{E}[Y^2] = 164 \quad \mathsf{E}[X \cdot Y] = 1,200$$

a. (5) Are X and Y independent? Why?

b. (5) $\mathsf{P}[75 \leq X \leq 150] =$ __________________

c. (5) $\mathsf{Var}[X - Y] =$ __________________

d. (5) $\mathsf{P}[X + Y \leq 121] =$ __________________

Problem 8: Will Scarlett decides it's too hard to make enough arrows to replace the ones they lose; he decides to catch them. Not a good idea.

Each time Will tries to catch an arrow, three things can happen: He catches it (with probability 1/2), he drops it (with probability 1/3), or he gets hurt (with probability 1/6).

a. (5) Find the probabilty that Will catches the first 10 arrows in a row (don't worry about the 11^{th} arrow).

b. (5) What is the probability distribution for the total number of arrows (dropped or caught) *before* Will gets hurt? Give its name and its mean.

c. (5) Give an expression for the probability that Will catches exactly k arrows before he gets hurt (he might drop some along the way). Bonus (+2) for the simplest possible expression.

d. (5) Want a five-point bonus? Check here: ◯

Problem 9: Robin Hood climbs the vines outside the castle, hoping to reach Maid Marian's window 20 steps up.

Robin has prob. $p_1 = \frac{1}{2}$ of reaching the first step successfully, $p_2 = \frac{2}{3}$ of reaching the second, and in general prob. $p_n = \frac{n}{n+1}$ of reaching the n^{th} step successfully— and prob. $q_n = [1-p_n] = \frac{1}{n+1}$ of falling all the way down to the ground. Whenever he falls, he starts again. People in love do silly things.

a. (5) What is the probability that he reaches the window without falling?

$\mathsf{P}[\text{no fall}] =$ ____________________

b. (5) Find the probability distribution for the number of falls Y he suffers, before reaching Marian's window. What is its mean?

c. (5) Find the probability distribution for the number of successful steps X he takes before his first fall (or reaching Marian's window, whichever comes first). For example, $\mathsf{P}[X = 0] = q_1 = \frac{1}{2}$, and $\mathsf{P}[X = 8] = \frac{1}{90}$.

d. (5) Find a simple expression (sum or integral) for the expected value of X (you need not evaluate it; numerically it should be about 2.645):

$\mathsf{E}[X] =$ ____________________

Problem 10: The length of time Robin takes to run one furlong is a continuously-distributed random variable with the uniform distribution on the interval from 55 to 65 seconds. Times for different furlongs are independent.

a. (5) Find the (exact) probability that Robin runs the first furlong within ± 1 second of his average time, one minute:

$\mathsf{P}[|T_1 - 1\,\text{min}| \le 1\,\text{sec}] =$ ____________________

b. (5) Find the (approximate) probability that Robin runs the first 48 furlongs within ± 1 second of his average time, 48 minutes:

$\mathsf{P}[|T_{48} - 48\,\text{mins}| \le 1\,\text{sec}] =$ ____________________

c. (5) Find numbers a and b such that, for large n, the approximate probability that Robin runs the first n furlongs within ± 1 second of his average time, n minutes, is about a/n^b:

$\mathsf{P}[|T_n - n\,\text{min}| \le 1\,\text{sec}] \approx \dfrac{a}{n^b}$ for $a =$ _______________ $b =$ _______

d. (5) The mean of T_n is n minutes, yet $\mathsf{P}[|T_n - n\,\text{min}| \le 1\,\text{sec}] \to 0$ as $n \to \infty$. Why doesn't this violate the "law of large numbers"?

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

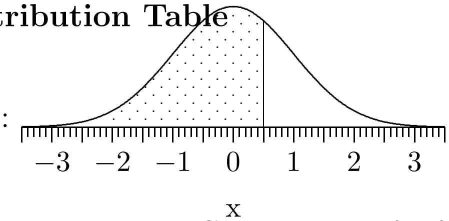

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n, p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\, q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n, A, B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu, \beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu, \sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha, p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha, \beta)$	$f(x) = \beta\, \alpha^\beta / x^{\beta+1}$	$x \in (\alpha, \infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2}\, x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a, b)$	$f(x) = \frac{1}{b-a}$	$x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha, \beta, \gamma)$	$f(x) = \frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha} e^{-[(x-\gamma)/\beta]^\alpha}$	$x \in (\gamma, \infty)$	$\gamma + \beta\Gamma(1+\alpha^{-1})$		