Final Examination (Ver A)

Mth 135 = Sta 104

Tuesday, 2003 December 9, 9:00am – 12:00n

This is a **closed-book** exam. You may use a calculator and a two-sided single sheet of *your own* notes, if you wish, but you may not share materials. A normal distribution table, a page of common pdf/pmf formulas, and a blank sheet are attached. If you don't understand something in one of the questions **please** ask me.

Each problem should take about 10–15 minutes. All problems count equally, even though they are not equally difficult. Point values for problem parts are indicated in parentheses.

You must **show** your **work** to get credit. Unsupported answers aren't acceptable, even if they're correct. Please give all numerical answers as fractions **in lowest terms** (simplify!) or as decimals correct to **four places**. It is to your advantage to write your solutions as clearly as possible.

Cheating on exams is a breach of trust with classmates and faculty, and will not be tolerated. Please acknowledge the Duke Community Standard:

I have adhered to the Duke Community Standard
in completing this examination.

Print: ______________________ Sign: ______________________

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
Total:			/200

Problem 1: Bozo the Clown is taking a refresher course at Clown College, and must take a multiple-choice test. Each question has 5 possible answers, one correct and four incorrect. Bozo knows the correct answer for about 80% of this sort of question. For each question he will answer correctly, if he knows the answer; and otherwise will guess, with each possible answer equally likely.

a) (5) What is the probability that Bozo will answer the first question correctly?

$\mathsf{P}[\text{Prob 1 Correct}] =$ ____________

b) (5) If Bozo *does* answer Problem 1 correctly, what is the probability that he actually knows the answer?

$\mathsf{P}[\text{Knows answer}|\text{Prob 1 Correct}] =$ ____________

c) (6) Suppose that there are 25 questions on the exam. Let N be the number that Bozo answers correctly. Find the mean and variance:

$\mathsf{E}[N] =$ ____________ $\mathsf{Var}[N] =$ ____________

d) (4) What is the *exact* probability distribution of N? Give its name and the values of any parameter(s). Half-credit for an *approximate* distribution with parameter(s). Make any assumptions you feel are needed.

Problem 2: Choose the best probability distribution for each random variable below from among the choices *Beta*, *Binomial*, *Exponential*, *Gamma*, *Geometric*, *Hypergeometric*, *Negative Binomial*, *Normal*, *Poisson*, or *Uniform* and, whatever the distribution, give its mean μ:

a) (4) The number of times Bozo's squirting flower works correctly before it fails, if failures are independent and have probability 0.01:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

b) (4) The number of shoes with holes in the soles, from a random draw of 6 shoes (without replacement) from Bozo's collection of 20 shoes, 8 of which have holes:

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

c) (4) The number of children in Row 6 who mistake Bozo for Ronald McDonald, if 25% of children make that mistake and if Row 6 holds 12 children today?

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

d) (4) The number of times Bozo laughs in an hour, if he laughs about once every five minutes and if the numbers of laughs in different periods are independent?

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

e) (4) The total amount of weight Bozo gains in his 200 working days a year, if daily gains are independent and average 0.01 *lb* per day?

◯ Be ◯ Bi ◯ Ex ◯ Ga ◯ Ge ◯ HG ◯ NB ◯ No ◯ Po ◯ Un

$\mu =$

Problem 3: After his performance Bozo passes a hat for tips. The hat garners \$34 in ten US bills: four one-dollar bills and six five-dollar bills. Bozo pulls three bills from the hat at random. Let X denote the denomination of the smallest bill drawn and let Y denote the denomination of the largest bill drawn; for example, if the three bills were $1, 5, 1$, then $X = 1$ and $Y = 5$.

a) (5) Find the joint probability mass function for X and Y:

$$p(x, y) = \begin{cases} \\ \\ \end{cases}$$

b) (5) Find the marginal probability mass functions:

$$p_X(x) = \begin{cases} \\ \\ \end{cases} \qquad p_Y(y) = \begin{cases} \\ \\ \end{cases}$$

c) (5) Find the conditional probability mass function:

$$p(y|X = 1) = \begin{cases} \\ \\ \end{cases}$$

d) (5) Are X and Y independent? Explain.

Problem 4: Felix Adler was a real clown in the Ringling Brothers Circus who trained piglets to slide down a greased slide. The piglets, all named Amelia (after his wife), were given away when they grew too big; over the years there were 360 Amelia's. Suppose that piglet weights are independent and normally distributed, with mean 10 *lb* and variance 1.6 lb^2.

a) (5) What is the probability that all Felix's piglets together weigh more than a 3650 *lb* Ferrarri?

$\mathsf{P}[S > 3650] =$ ________________

b) (5) What is the probability that the *maximum* weight of Felix's 360 piglets is less than 14 *lb*?

$\mathsf{P}[\max X_i \leq 14] =$ ________________

c) (5) About how many of Felix's 360 piglets weigh *less* than 8.75 *lb*?

$\mathsf{E}[\text{Number of piglets weighing } X_i \leq 8.75] =$ ________________

d) (5) ◯ Check here for 5 free points. This problem was getting too long.

Problem 5: Cooky the Clown offers to play a game with you. First, you pay Cooky \$10. Then, Cooky pays you back an amount that depends on the value of a random variable Z. She draws Z from the $\mathsf{Ex}(1)$ distribution with pdf and CDF

$$f(z) = \begin{cases} e^{-z} & z > 0 \\ 0 & z \le 0 \end{cases} \qquad F(z) = \begin{cases} 1 - e^{-z} & z > 0 \\ 0 & z \le 0 \end{cases}$$

a) (5) If Cooky pays you $A = 10Z$, find your expected return and your probability of winning:

$\mathsf{E}[A] =$ __________ $\mathsf{P}[A > 10] =$ __________

b) (5) If Cooky pays you $B = Z^2$, find: (Hint: You can do this without integrating, if you know the mean *and variance* of Z).

$\mathsf{E}[B] =$ __________ $\mathsf{P}[B > 10] =$ __________

c) (5) If Cooky pays you $C = e^Z$, find:

$\mathsf{E}[C] =$ __________ $\mathsf{P}[C > 10] =$ __________

d) (5) You get to choose a number T; Cookie pays you $D = \$T$ if $Z > T$, or $D = \$0$ if $Z \le T$. What value of T maximizes your expected return $\mathsf{E}[D]$? For that T, what is $\mathsf{E}[D]$?

Best $T =$ __________ $\mathsf{E}[D] =$ __________

Problem 6: Pierrot and Harlequin are traditional French clown characters, sometimes portrayed as musicians. Our friends P and H have a (non-lethal) confetti bomb whose fuse lasts a random time T (not necessarily positive!) with mean $\mathsf{E}[T] = 1$ and variance $\mathsf{Var}[T] = 1$. Unfortunately neither of them read the instructions to find out what distribution T has. Sigh.

a) (4) Find $\mathsf{P}[T \geq 2]$ if T has an exponential distribution.

b) (4) Find $\mathsf{P}[T \geq 2]$ if T has a Poisson distribution.

c) (4) Find $\mathsf{P}[T \geq 2]$ if T has a normal distribution.

d) (4) Find $\mathsf{P}[T \geq 2]$ if T has a uniform distribution.

e) (4) Find an upper bound for $\mathsf{P}[T \geq 2]$ for *all* positive random variables T with mean $\mathsf{E}[T] = 1$. Note that positivity is assumed only for e).

Problem 7: Each day in the mail Krusty the Clown gets a new pair of shoes. The left and right shoe-sizes X and Y are independent normally distributed random variables, with means $\mathsf{E}[X] = 13$ and $\mathsf{E}[Y] = 15$ and variances $\mathsf{Var}[X] = 9$ and $\mathsf{Var}[Y] = 16$. Find:

a) (4) $\mathsf{P}[X > 10] =$ ______________

b) (4) $\mathsf{P}[X < Y] =$ ______________

c) (4) $\mathsf{E}[X \cdot Y] =$ ______________

d) (4) $\mathsf{Var}[X - Y] =$ ______________

e) (4) $\mathsf{P}[X < 14.5 | Y = 13] =$ ______________

Problem 8: Buttons the Clown uses a lot of makeup. The total volume of face paint Buttons uses in a week (in ounces) is a random variable X with an exponential distribution with mean $\mathsf{E}[X] = 2$. Weeks are independent.

a) (5) Give the p.d.f. for X, the weekly volume of face paint.

$$f_X(x) = \begin{cases} \\ \end{cases}$$

b) (5) Find the p.d.f. for $Y = X^{2/3}$, the area (in square inches) Buttons can cover with X ounces of face paint.

$$f_Y(y) = \begin{cases} \\ \end{cases}$$

c) (5) Find the probabilty that Buttons uses more than 1 ounce on Week 1, more than 2 ounces on Week 2, and more than 3 ounces on Week 3:

$\mathsf{P}[(X_1 > 1) \cap (X_2 > 2) \cap (X_3 > 3)] =$ ________________

d) (5) Find the probability that Buttons uses an increasing amount of face paint in weeks 10–12: (Hint: This is easy)

$\mathsf{P}[X_{10} < X_{11} < X_{12}] =$ ________________

Problem 9: The joint probability density function for the random variables X and Y is given by

$$f(x,y) = \begin{cases} 4\,e^{-2y} & 0 < x \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

a) (6) Find the marginal probability density function for X, identify it by name, and give its mean: $f_X(x) =$ ____________________;

○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un

$\mu =$

b) (6) Find the marginal pdf for Y, identify it by name, and give its mean:

$f_Y(y) =$ ____________________;

○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un

$\mu =$

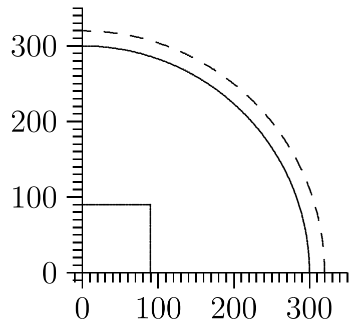
c) (6) Find the *conditional* pdf for X, given $Y = 4$; identify it by name, and give its mean: $f_{X|Y}(x \mid 4) =$ ____________________;

○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un

$\mu =$

d) (2) Bonus time again!

Problem 10: Krusty the Clown is up at bat in the annual Springfield Clowns Against Cancer benefit baseball game. The field is shaped like a quarter-circle of radius $300'$ (the distance from home plate to the outfield fence, in feet). Whenever Krusty bats, the ball hits the ground first at a point distributed uniformly in a quarter-circle of radius $320'$ (to the dotted line in the figure).



a) (6) What is the probability that Krusty hits the ball into the infield, a square of dimensions $90' \times 90'$?

b) (6) What is the probability that Krusty hits a home run over the outfield fence?

c) (8) What is the probability density function for the distance R Krusty hits the ball?

$$f(r) = \Bigg\{$$

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

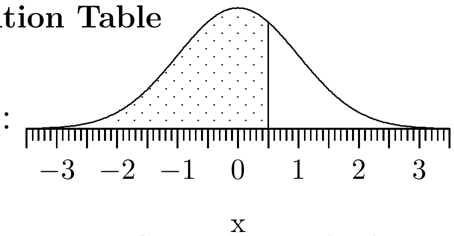


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\, q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(x) = p\, q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(x) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x) = \beta\,\alpha^\beta / x^{\beta+1}$	$x \in (\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times$ $x^{\frac{\nu_1-2}{2}} \left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} [1+x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x) = \frac{1}{b-a}$	$x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x) = \frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha} e^{-[(x-\gamma)/\beta]^\alpha}$	$x \in (\gamma,\infty)$	$\gamma + \beta\Gamma(1+\alpha^{-1})$		