9.0 Lesson Plan

- Discuss Quizzes/Answer Questions
- History Note
- Review
- Permutations and Combinations
- Binomial Probability

9.1 History Note

Pascal and Fermat laid out the basic rules of probability in a series of letters. One of the most famous consequences of that discussion was Pascal's Wager.

Pascal's Wager defined two events:

- $A = \{ \text{ God exists } \}$
- $A^c = \{ \text{ God does not exist } \}$

A^c is the complement of A, which is the event that A does not happen. From Kolmogorov's Axioms, it is easy to see that for any event A,

$$P[A^c] = 1 - P[A].$$

Pascal also defined a payoff matrix:

	Bet on A	**Bet on A^c**
A true	**∞ reward**	**∞ loss**
A^c true	**finite misery**	**finite reward**

Pascal argued that in terms of gambling, the optimal strategy for this game is to believe in God, no matter how small the (Bayesian subjective) value for $P[A]$ might be.

Modern game theory does not quite agree with Pascal, but the issues are rather technical; e.g., infinite payoffs are problematic.

9.2 Review

Recall Kolmogorov's Axioms:

- $0 \leq P[A] \leq 1$
- $P[$ some possible event happens $] = 1$ (one of the possible outcomes must occur).
- If A and B are incompatible events, then $P[A \text{ or } B] = P[A] + P[B]$.

From this, note that $P[A^c] = 1 - P[A]$. Why?

Also recall the rules of conditional probability:

$$P[A|B] = \frac{P[A \textbf{ and } B]}{P[B]}; \qquad P[A \textbf{ and } B] = P[A|B] * P[B].$$

Consider one draw from a deck of cards. Let $A = \{$ draw a king $\}$ and $B = \{$ draw a queen $\}$. What is $P[A \text{ or } B]$?

These are mutually exclusive events, so

$$P[A \textbf{ or } B] = P[A] + P[B] = 4/52 + 4/52 = 2/13.$$

Consider two draws from a deck, without replacement. Let $A = \{$ second a king $\}$ and $B = \{$ first a queen $\}$. What is $P[A \text{ and } B]$?

$$P[A \textbf{ and } B] = P[A|B] * P[B] = 4/51 + 4/52 = .0060331.$$

Recall that A and B are independent if $P[A|B] = P[A] * P[B]$. This implies that $P[B|A] = P[A] * P[B]$. Why?

It also implies that A and B are independent iff $P[A \text{ and } B] = P[A] * P[B]$. Why?

What is the probability of rolling first a 1, then a 2, then a 3 on consecutive rolls of a fair die? Since rolls are independent,

$$\begin{aligned} P[\ \textbf{1 on first, 2 on second, 3 on third}\ ] &= P[\ \textbf{1 on first}\ ] * \\ & \quad P[\ \textbf{2 on second}\ ] * \\ & \quad P[\ \textbf{3 on third}\ ] \\ &= 1/6 * 1/6 * 1/6 \\ &= 1/216. \end{aligned}$$

What is the probability of rolling a 1, 2, and 3 in any order in three consecutive rolls of a fair die?

$$\begin{aligned} P[\ \mathbf{1,\ 2,\ 3\ in\ any\ order}\ ] &= P[(1,2,3)\ \mathbf{or}\ (2,1,3)\ \mathbf{or}\ \cdots\ \mathbf{or}\ (3,2,1)] \\ &= P[(1,2,3)] + P[(2,1,3)] + \cdots + P[(3,2,1)] \\ &= 6 \times (\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}) \end{aligned}$$

which is just 1/36.

The first step lists all six possible orders, the second step uses the fact that different orders are mutually exclusive, and the third uses independence and the fact that die is fair.

Another way to get the same answer is to note that
$P[$ 1, 2, 3 in any order $]$ is

$$\begin{aligned} P[\,\textbf{1, 2, or 3 on first}\,] \quad \times \\ P[\,\textbf{one of remaining two on second} \mid \textbf{result of first roll}\,] \quad \times \\ P[\,\textbf{single remaining number} \mid \textbf{result of first two rolls}\,] \quad = \\ \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6} \end{aligned}$$

and this is also 1/36.

Note that in order to get this solution, one had to be able to count all the ways to rearrange the numbers (1, 2, 3). These different arrangements are called permutations.

Another useful trick is to use complementary events. For example, what is the probability of getting at least on 6 in three consecutive rolls?

$$
\begin{aligned}
P[\ \textbf{at least one 6 in three rolls}\ ] &= \\
1 - P[\ \textbf{no 6 in three rolls}\ ] &= \\
1 - P[\ \textbf{no 6 on first; no 6 on second; no 6 on third}\ ] &= \\
1 - P[\ \textbf{no 6 on first}\ ] \times P[\ \textbf{no 6 on second}\ ] \times P[\ \textbf{no 6 on third}\ ] &= \\
1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} &= \\
\frac{91}{216}.
\end{aligned}
$$

9.3 Permutations and Combinations

The number of ways to arrange n distinct objects in a line is $n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$

Here $n!$ is called n factorial and it is the number of permutations of n distinct objects.

There are n choices for the first position, then $n-1$ for the second, and so forth; multiplication gives the number of distinct arrangments.

By convention, $0! = 1$. For the other positive integers,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \ldots$$

In how many ways can 8 people line up? $8! = 40320$.

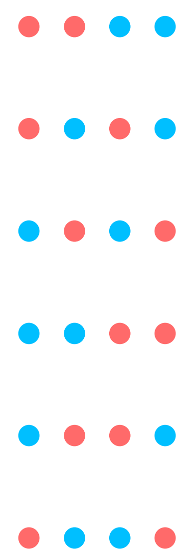
In how many ways can 4 married couples stand in a police line-up, if couples must stand together?

There are $4!$=24 ways that the couples can be arranged, and each couple can be arranged in $2! = 2$ ways. So the answer is

$$(4!) \times (2!) \times (2!) \times (2!) \times (2!) = 384.$$

The number of ways to arrange r red marbles and $n-r$ blue marbles in a line is: $\binom{n}{r} = \frac{n!}{r! \times (n-r)!}$

For example, the number of ways to arrange two red marbles and two green marbles is:



This is just $\binom{4}{2} = \frac{4!}{2! \times 2!} = \frac{24}{2 \times 2} = 6.$

Note: The number $\binom{n}{r}$ is also the number of ways to pick r objects from a set of n distinct objects. Why?

When she was four years old, my daughter had 20 stuffed animals. Her bed could only hold 7. Being perfectly promiscuous, she chose a different set each night. How long before she must repeat a set?

$$\binom{20}{7} = \frac{20!}{7! \times 13!} = 77520.$$

This is about 212 years.

9.4 Binomial Probability

The binomial formula gives the probability of exactly r successes in n tries, where each try has the same prbability of success p and each try is independent.

$$P[\text{ \textbf{exactly} } r \text{ \textbf{successes} }] = \binom{n}{r} p^r (1-p)^{n-r}.$$

This is exactly the situation one has when trying to find the probability of r heads in n tosses of a coin that has probability p of coming up heads.

Why does this formula work?

1. How many arrangements are there that give r heads in n tries? From the previous section, we know the answer is $\begin{pmatrix} n \\ r \end{pmatrix}$.

2. Each arrangement is incompatible with the other arrangements. Thus the probability of exactly r successes is the sum over all possible arrangements, and there are $\begin{pmatrix} n \\ r \end{pmatrix}$ of those.

3. Each arrangement has the same probability: $p^r(1-p)^{n-r}$. To see this, consider the sequences HTHT and THTH. The first has probability $p*(1-p)*p(1-p) = p^2(1-p)^2$. The second arrangement has probability $(1-p)*p*(1-p)*p = p^2(1-p)^2$.

Thus the binomial formula is

$$P[\text{ \textbf{exactly} } r \text{ \textbf{successes} }] = \left(\begin{array}{c} n \\ r \end{array} \right) p^r (1-p)^{n-r}.$$

Find the probability of exactly two sixes in five rolls of a fair die.

$$\begin{aligned} P[\text{ \textbf{exactly} } 2 \text{ \textbf{successes} }] &= \left(\begin{array}{c} n \\ r \end{array} \right) p^r (1-p)^{n-r} \\ &= \left(\begin{array}{c} 5 \\ 2 \end{array} \right) (\frac{1}{6})^2 (1 - \frac{1}{6})^{5-2} \\ &= 10 \times (\frac{1}{6})^2 \times (\frac{5}{6})^3 \\ &= .16075. \end{aligned}$$